SEQUENCE TRANSFORMS RELATED TO REPRESENTATIONS USING GENERALIZED FIBONACCI NUMBERS

V. E. HOGGATT, JR. (Deceased)
and
MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, CA 95192

1. INTRODUCTION

We make use of the sequences $A = \{a_n\}$ and $B = \{b_n\}$, where (a_n, b_n) are safe-pairs in Wythoff's game, described by Ball [1], and, more recently, by Horadam [2], Silber [3], and Hoggatt & Hillman [4] to develop properties of sequences whose subscripts are given by a_n and b_n.

Let $U = \{u_i\}_{i=1}^{\infty}$. We define A and B transforms by

$$\begin{aligned} AU &= \{u_{a_i}\}_{i=1}^{\infty} = \{u_1, u_3, u_4, u_6, \ldots, u_{a_i}, \ldots\}, \\ BU &= \{u_{b_i}\}_{i=1}^{\infty} = \{u_2, u_5, u_7, \ldots, u_{b_i}, \ldots\}. \end{aligned} \tag{1.1}$$

Notice that, for $N = \{n_i\}$, $n_i = i$, the set of natural numbers, we have

$$AN = \{n_{a_i}\} = \{a_i\} = A,$$

$$BN = \{n_{b_i}\} = \{b_i\} = B.$$

Next, we list the first fifteen Wythoff pairs, and some of their properties which will be needed.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n:	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
b_n:	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39

Notice that we begin with $a_1 = 1$, and a_k is always the smallest integer not yet used. We find $b_n = a_n + n$. We list the following properties:

$$a_k + k = b_k \tag{1.2}$$

$$a_n + b_n = a_{b_n} \tag{1.3}$$

$$a_{a_n} + 1 = b_n \tag{1.4}$$

$$a_{k+1} - a_k = \begin{cases} 2, & k = a_n \\ 1, & k = b_n \end{cases} \tag{1.5}$$

$$b_{k+1} - b_k = \begin{cases} 3, & k = a_n \\ 2, & k = b_n \end{cases} \tag{1.6}$$

Further, (a_n, b_n) are related to the Fibonacci numbers in several ways, one being that, if $A = \{a_n\}$ and $B = \{b_n\}$, then A and B are the sets of positive integers for which the smallest Fibonacci number used in the unique Zeckendorf representation occurred respectively with an even or odd subscript [6].

2. A AND B TRANSFORMS OF A SPECIAL SET U (FIBONACCI CASE)

Let $U = \{u_i\}$, where

$$u_{m+1} - u_m = \begin{cases} p, & \text{if } m = a_k \\ q, & \text{if } m = b_k \end{cases} \tag{2.1}$$

Actually, we can write an explicit formula for u_m in terms of u_1, p, and q, as in the following theorem.

THEOREM 2.1: $u_m = (2m - 1 - a_m)q + (a_m - m)p + u_1$.

PROOF: $u_m = (u_m - u_{m-1}) + (u_{m-1} - u_{m-2}) + (u_{m-2} - u_{m-3}) + \cdots$

$$+ (u_3 - u_2) + (u_2 - u_1) + u_1$$

$$= (\text{no. of } b_j\text{'s less than } m)q + (\text{no. of } a_j\text{'s less than } m)p + u_1$$

$$= (2m - 1 - a_m)q + (a_m - m)p + u_1$$

by the following lemma.

LEMMA 1: The number of b_j's less than n is $(2n - 1 - a_n)$, and the number of a_j's less than n is $(a_n - n)$.

PROOF:

a_n:	1	3	4	6	8	9
n:	1	2	3	4	5	6
$a_n - n$:	0	1	1	2	3	3
a_j's less than n:	0	1	1	2	3	3

Notice that the lemma holds for $n = 1, 2, \ldots, 6$. Assume that the number of a_j's less than k is given by $a_k - k$. Then the number of a_j's less than $(k + 1)$ has to be either $(a_k - k)$ or $(a_k - k) + 1$. If $k = b_i$, then

$$a_{k+1} - (k + 1) = a_k + 1 - (k + 1) = a_k - k$$

by (1.5), while if $k = a_i$, then

$$a_{k+1} - (k + 1) = a_k + 2 - (k + 1) = a_k - k + 1,$$

giving the required result for $a_{k+1} - (k + 1)$. Thus, by mathematical induction, the number of a_j's less than n is given by $a_n - n$. But, the number of integers less than n is made up of the sum of the number of a_j's less than n and the number of b_j's less than n, since A and B are disjoint and cover the natural numbers. Thus,

$$n - 1 = (a_n - n) + (\text{number of } b_j\text{'s less than } n),$$

so that the number of b_j's less than n becomes $(2n - 1 - a_n)$.

We return to our sequence U and consider the A and B transforms. In particular, what are the differences of successive terms in the transformed sequences AU and BU?

For AU,

$$u_{a_{m+1}} - u_{a_m} = \begin{cases} q + p, & \text{if } m = a_k \\ p, & \text{if } m = b_k \end{cases} \tag{2.2}$$

Equation (2.2) is easy to establish by (1.5), since when $m = a_k$, $a_{m+1} = a_m + 2$, so that

$$u_{a_{m+1}} - u_{a_m} = (u_{a_m+2} - u_{a_m+1}) + (u_{a_m+1} - u_{a_m}) = (u_{b_i+1} - u_{b_i}) + p = q + p,$$

where we write $a_m + 1 = b_i$, because $a_m + 1 \neq a_k$ and A and B are disjoint and cover the natural numbers. For the second half of (2.2), since $a_{m+1} = a_m + 1$ by (1.5), we can apply (2.1) immediately.

For BU,

$$u_{b_{m+1}} - u_{b_m} = \begin{cases} 2p + q, & \text{if } m = a_k \\ p + q, & \text{if } m = b_k \end{cases} \tag{2.3}$$

We can establish (2.3) easily by (1.6), since when $m = a_k$, $b_{m+1} = b_m + 3$, and $b_m + 2 = a_i$, $b_m + 1 = a_j$ for some i and j, so we can write

$$\begin{aligned} u_{b_{m+1}} - u_{b_m} &= (u_{b_m+3} - u_{b_m+2}) + (u_{b_m+2} - u_{b_m+1}) + (u_{b_m+1} - u_{b_m}) \\ &= (u_{a_i+1} - u_{a_i}) + (u_{a_j+1} - u_{a_j}) + (u_{b_m+1} - u_{b_m}) \\ &= p + p + q = 2p + q. \end{aligned}$$

For the case $m = b_k$, $b_{m+1} = b_m + 2$ and $b_m + 1 = a_i$ for some i, causing

$$\begin{aligned} u_{b_{m+1}} - u_{b_m} &= (u_{b_m+2} - u_{b_m+1}) + (u_{b_m+1} - u_{b_m}) \\ &= (u_{a_i+1} - u_{a_i}) + (u_{b_m+1} - u_{b_m}) \\ &= p + q. \end{aligned}$$

Notice that we have put one b_i subscript on BU and one a_i subscript on AU. Now if we applied B twice, $BBU = \{u_{b_{b_i}}\}$ would have two successive b-subscripts, and we could record how many b-subscripts occurred by how many times we applied the B transform. Thus, a sequence of A and B transforms gives us a sequence of successive a- and b-subscripts. Further, we can easily handle this by matrix multiplication. Let the finally transformed sequence be denoted by $U^* = u^*_{(ab)_i}$ and define the difference of successive elements by

$$u^*_{(ab)_{i+1}} - u^*_{(ab)_i} = \begin{cases} p', & \text{if } i = a_k \\ q', & \text{if } i = b_k \end{cases}$$

and define the matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then AU has $p' = p + q$, $q' = p$, and

$$Q\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p + q \\ p \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix},$$

and BU has $p' = 2p + q$, $q' = p + q$, and

$$Q^2\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2p + q \\ p + q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

Now, the Q-matrix has the well-known and easily established formula

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for the Fibonacci numbers $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$.

Suppose we do a sequence of A and B transforms,

$$AABAAAU = A^2B^1A^3U.$$

Then the difference of successive terms, p' and q', are given by

$$Q^7\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} F_8 & F_7 \\ F_7 & F_6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} F_8p + F_7q \\ F_7p + F_6q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

Note that each A transform contributes Q^1 but a B transform contributes Q^2 to the product. Also, the sequence considered has successively 3 a-subscripts, one b-subscript, and 2 a-subscripts, so that $u^*_{(ab)_i}$ has six subscripted subscripts, or,

$$u^*_{(ab)_i} = u_{a_{a_{a_{b_{a_{a_i}}}}}}.$$

Also notice that the order of the A and B transforms does not matter. Thus, if U^* is formed after m A transforms and n B transforms in any order, then the matrix multiplier is Q^{m+2n}, and

$$p' = F_{m+2n+1}p + F_{m+2n}q, \qquad q' = F_{m+2n}p + F_{m+2n-1}q.$$

Comments on A and B Transforms

Let W be the weight of the sequence of A and B transforms, where each B is weighted 2 and each A weighted 1. Thus, the number of different sequences with weight W is the number of compositions of W using 1's and 2's, so that the number of distinct sequences of A and B transforms of weight W is F_{W+1}. Thus, u_1 in Theorem 2.1 can be any number 1, 2, ..., F_{W+1} for sequences of A and B transforms of weight W.

3. A, B, AND C TRANSFORMS (TRIBONACCI CASE)

The Tribonacci numbers T_n are

$$T_0 = 0,\ T_1 = 1,\ T_2 = 1,\ T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad n \geqslant 0.$$

Divide the positive integers into three disjoint subsets $A = \{A_k\}$, $B = \{B_k\}$, and $C = \{C_k\}$ by examining the smallest term T_k used in the unique Zeckendorf representation in terms of Tribonacci numbers. Let $n \in A$ if $k \equiv 2 \bmod 3$, $n \in B$ if $k \equiv 3 \bmod 3$, and $n \in C$ if $k \equiv 1 \bmod 3$. The numbers A_n, B_n, and C_n were considered in [6]. We list the first few values.

TABLE 3.1

n	A_n	B_n	C_n
1	1	2	4
2	3	6	11
3	5	9	17
4	7	13	24
5	8	15	28
6	10	19	35
7	12	22	41
8	14	26	48
9	16	30	55
10	18	33	61

Notice that we begin with $A_1 = 1$ and A_k is the smallest integer not yet used in building the array. Some basic properties are:

$$A_n + B_n + n = C_n \tag{3.1}$$

$$A_{A_n} + 1 = B_n, \qquad A_{B_n} + 1 = C_n \tag{3.2}$$

$$A_{n+1} - A_n = \begin{cases} 2, & n \in A \\ 2, & n \in B \\ 1, & n \in C \end{cases} \tag{3.3}$$

$$B_{n+1} - B_n = \begin{cases} 4, & n \in A \\ 3, & n \in B \\ 2, & n \in C \end{cases} \tag{3.4}$$

$$C_{n+1} - C_n = \begin{cases} 7, & n \in A \\ 6, & n \in B \\ 4, & n \in C \end{cases} \tag{3.5}$$

Let the special sequence $U = \{u_i\}$, where

$$u_{m+1} - y_m = \begin{cases} p, & m \in A \\ q, & m \in B \\ r, & m \in C \end{cases} \tag{3.6}$$

We can write an explicit formula for u_m in terms of u_1, p, q, and r.

<u>*THEOREM 3.1:*</u> $u_m = (2m - 1 - A_m)r + (2A_m - B_m)q + (B_m - A_m - m)p + u_1$.

<u>*PROOF:*</u> $u_m = (u_m - u_{m-1}) + (u_{m-1} - u_{m-2}) + \cdots + (u_3 - u_2) + (u_2 - u_1) + u_1$

$= (\text{no. of } C_j\text{'s less than } m)r + (\text{no. of } B\ \text{'s less than } m)q$

$+ (\text{no. of } A_j\text{'s less than } m)p + u_1.$

But, Theorem 4.5 of [6] gives $(2m - 1 - A_m)$ as the number of C_j's less than m, $(2A_m - B_m)$ as the number of B_j's less than m, and $(B_m - A_m - m)$ as the number of A_j's less than m, establishing Theorem 3.1.

We now return to our special sequence U of (3.6) and consider A, B, and C transforms as in Section 2. For AU,

$$u_{A_{m+1}} - u_{A_m} = \begin{cases} p + q, & m \in A \\ p + r, & m \in B \\ p, & m \in C \end{cases} \tag{3.7}$$

To establish (3.7), recall (3.3). If $m \in A$, then

$$\begin{aligned} u_{A_{m+1}} - u_{A_m} &= u_{A_m+2} - u_{A_m+1} + u_{A_m+1} - u_{A_m} \\ &= u_{B_n+1} - u_{B_n} + u_{A_m+1} - u_{A_m} \\ &= q + p. \end{aligned}$$

If $m \in B$,

$$\begin{aligned} u_{A_{m+1}} - u_{A_m} &= u_{A_m+2} - u_{A_m+1} + u_{A_m+1} - u_{A_m} \\ &= u_{C_n+1} - u_{C_n} + u_{A_m+1} - u_{A_m} \\ &= r + p. \end{aligned}$$

If $m \in C$,

$$u_{A_{m+1}} - u_{A_m} = u_{A_m + 1} - u_{A_m} = p.$$

Now, matrix T,

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

can be used to write AU, since

$$T \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p + q \\ p + r \\ p \end{pmatrix}. \tag{3.8}$$

Notice that the characteristic polynomial of T is $x^3 - x^2 - x - 1 = 0$, while the characteristic polynomial of Q of Section 2 is $x^2 - x - 1 = 0$.

In an entirely similar manner, for BU one can establish

$$u_{B_{m+1}} - u_{B_m} = \begin{cases} 2p + q + r, & m \in A \\ 2p + q, & m \in B \\ p + q, & m \in C \end{cases} \tag{3.9}$$

and for CU,

$$u_{C_{m+1}} - u_{C_m} = \begin{cases} 4p + 2q + r, & m \in A \\ 3p + 2q + r, & m \in B \\ 2p + q + r, & m \in C \end{cases} \tag{3.10}$$

We compute BU as

$$T^2 \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2p + q + r \\ 2p + q \\ p + q \end{pmatrix}$$

and CU as

$$T^3 \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 4p + 2q + r \\ 3p + 2q + r \\ 2p + q + r \end{pmatrix}.$$

We note that

$$T^n = \begin{pmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n + T_{n-1} & T_{n-1} + T_{n-2} & T_{n-2} + T_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{pmatrix}, \tag{3.11}$$

which could be proved by mathematical induction.

We may now apply A, B, and C transforms in sequences. If we assign 1 as weight for A, 2 as weight for B, and 3 as weight for C, then there are T_{n+1} sequences of A, B, and C of weight n corresponding to the compositions of n in terms of 1's, 2's, and 3's. Since any positive integer in sequence A_n, B_n, or C_n can be brought to u_1 by a unique sequence of A, B, or C transforms, there is a unique correspondence between the positive integers and the compositions of n in terms of 1's, 2's, and 3's.

4. A, B, AND C TRANSFORMS OF THE SECOND KIND

We now consider the sequence defined by

$$U_1 = 1,\ U_2 = 2,\ U_3 = 3,\ U_{n+3} = U_{n+2} + U_n,$$

with characteristic polynomial $x^3 - x - 1 = 0$. We define $A = \{A_n\}$, $B = \{B_n\}$, $C = \{C_n\}$, and let $H = \{H_n\}$ be the complement of $B = A \cup C$, where A, B, and C are disjoint and cover the set of positive integers, as follows:

$$\begin{aligned} A_n &= \text{smallest positive integer not yet used} \\ B_n &= A_n + n \\ C_n &= B_n + H_n = A_n + B_n - (\text{number of } C_j\text{'s less than } A_n) \end{aligned} \tag{4.1}$$

This array has many interesting properties [6], [8], but here the main theme is the representations in terms of the sequence U_n above. We list the first terms in the array for n, A, B, C, and H in the following table.

TABLE 4.1

n	A_n	B_n	H_n	C_n
1	1	2	1	3
2	4	6	3	9
3	5	8	4	12
4	7	11	5	16
5	10	15	7	22
6	13	19	9	28

Here we can also obtain sets A, B, and C by examining the smallest term U_k used in the unique Zeckendorf representation of an integer N in terms of the sequence U_k. We let $N \in A$ if $k \equiv 1 \bmod 3$, $N \in B$ if $k \equiv 2 \bmod 3$, and $N \in C$ if $k \equiv 3 \bmod 3$.

From Theorem 7.4 of [6], we have:

$$A_{n+1} - A_n = \begin{cases} 3, & n = A_k \\ 1, & n = B_k \\ 2, & n = C_k \end{cases} \tag{4.2}$$

$$B_{n+1} - B_n = \begin{cases} 4, & n = A_k \\ 2, & n = B_k \\ 3, & n = C_k \end{cases} \tag{4.3}$$

$$C_{n+1} - C_n = \begin{cases} 6, & n = A_k \\ 3, & n = B_k \\ 4, & n = C_k \end{cases} \tag{4.4}$$

Let the special sequence $U = \{u_i\}$, where

$$u_{m+1} - u_m = \begin{cases} p, & m \varepsilon A \\ q, & m \varepsilon B \\ r, & m \varepsilon C \end{cases} \tag{4.5}$$

We can now write an explicit formula for u_m in terms of u_1, p, q, and r.

THEOREM 4.1: $u_m = (C_m - B_m - m)p + (C_m - 2A_m - 1)q + (3B_m - 2C_m)r + u_1$.

PROOF: $u_m = (u_m - u_{m-1}) + (u_{m-1} - u_{m-2}) + \cdots + (u_3 - u_2) + (u_2 - u_1) + u_1$

$= (\text{no. of } A_j\text{'s less than } m)p + (\text{no. of } B_j\text{'s less than } m)q$

$+ (\text{no. of } C_j\text{'s less than } m)r + u_1.$

Corollary 7.4.1 of [6] gives the number of A_j's less than m as $C_m - B_m - m$, the number of B_j's less than m as $C_m - 2A_m - 1$, and the number of C_j's less than m as $3B_m - 2C_m$. Each of these is zero for $m = 1$.

We again return to our special sequence U of (4.5) and consider A, B, and C transforms as in Section 2. We write the matrix Q^* and consider the AU, BU, and CU transforms:

$$Q^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For AU, we have

$$Q^{*2}V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p + q + r \\ p \\ p + q \end{pmatrix}$$

and

$$u_{A_{m+1}} - u_{A_m} = \begin{cases} p + q + r, & m \varepsilon A \\ p, & m \varepsilon B \\ p + q, & m \varepsilon C \end{cases} \tag{4.6}$$

For BU, we write the matrix multiplication $Q^{*3}V$,

$$Q^{*3}V = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2p + q + r \\ p + q \\ p + q + r \end{pmatrix}$$

and

$$u_{B_{m+1}} - u_{B_m} = \begin{cases} 2p + q + r, & m \in A; \\ p + q, & m \in B; \\ p + q + r, & m \in C. \end{cases} \tag{4.7}$$

For CU, we write $Q^{*4}V$,

$$Q^{*4}V = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3p + 2q + r \\ p + q + r \\ 2p + q + r \end{pmatrix}$$

and

$$u_{C_{m+1}} - u_{C_m} = \begin{cases} 3p + 2q + r, & m \in A; \\ p + q + r, & m \in B; \\ 2p + q + r, & m \in C. \end{cases} \tag{4.8}$$

Here, as a bonus, we can work with the transformation HU by using the matrix Q^* itself. Since $A \cup B \cup C = N$, using H and B transforms corresponds to the number of compositions of n using 1's and 3's, which is given in terms of the sequence U_n, defined at the beginning of this section by U_{n-1}.

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