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THE PARITY OF THE CATALAN NUMBERS VIA LATTICE PATHS

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The Catalan numbers

$$C_n = \binom{2n}{n} / (n + 1)$$

belong to the class of advanced counting numbers that appear as naturally and almost as frequently as the binomial coefficients, due to the extensive variety of combinatorial objects counted by them (see [1], [2]).

The purpose of this note is to give a combinatorial proof of the following property of the Catalan sequence using a lattice path interpretation.

Theorem

C_n is odd if and only if $n = 2^r - 1$ for some positive integer r.

Proof: The proof is based mainly on the following observation: If X is a finite set and α is an involution on X with fixed point set X^α, then $|X| \equiv |X^\alpha| \pmod 2$; i.e., $|X|$ and $|X^\alpha|$ have the same parity.

Now let D_n denote the set of lattice paths in the first quadrant from the origin to the point $(2n, 0)$ with the elementary steps

$$x: (a, b) \to (a + 1, b + 1)$$

$$\bar{x}: (a, b) \to (a + 1, b - 1).$$

It is well known that $|D_n| = C_n$ (see [2],[3]). Define $\alpha: D_n \to D_n$ by reflecting these paths about the line $x = n$. The fixed point set D_n^α of α consists of all paths in D_n symmetric with respect to the line $x = n$.

Now define an involution β on D_n^α as follows: for $w = w_1 u \bar{u} w_2 \in D_n^\alpha$ with $|w_1| = |w_2| = n - 1$ and $u \in \{x, \bar{x}\}$, set

$$\beta(w) = \begin{cases} w_1 \bar{u} u w_2 & \text{if } w_1 \notin D_{\frac{n-1}{2}} \\ w & \text{otherwise.} \end{cases}$$

Of course the set $D_{\frac{n-1}{2}}$ is empty unless n is odd. Hence, we can put

$$C_{\frac{n-1}{2}} = 0 \text{ for } n \text{ even.}$$

Note that

$$|D_n^{\alpha\beta}| = \left|D_{\frac{n-1}{2}}\right|,$$

since $w \to w_1$ is an obvious bijection between the sets $D_n^{\alpha\beta}$ and $D_{\frac{n-1}{2}}$. Thus we have

$$C_n \equiv C_{\frac{n-1}{2}} \pmod 2. \tag{1}$$

If C_n is odd, then induction on n gives $(n - 1)/2 = 2^r - 1$ for some r so that $n = 2^{r+1} - 1$ is of the required form. Of course, $C_{2-1} = C_1 = 1$.

The converse also follows immediately from (1) by a similar inductive argument.

REFERENCES

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