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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-490 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that the arithmetic mean of $L_{2n}L_{2n+3}$ and $5F_{2n}F_{2n+3}$ is always a Lucas number.

B-491 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_kF_{n+j} - F_jF_{n+k} = (L_jL_{n+k} - L_kL_{n+j})/5.$$

B-492 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_nF_{n+j+k} - F_{n+j}F_{n+k} = (L_{n+j}L_{n+k} - L_nL_{n+j+k})/5.$$

B-493 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Derive a formula for the largest integer $e = e(n)$ such that 2^e is an integral divisor of

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i},$$

where $\binom{n}{k} = 0$ for $k > n$.

B-494 *Proposed by Philip L. Mana, Albuquerque, NM*

For each positive integer n, find positive integers a_n and b_n such that $101n$ is the following sum of consecutive positive integers:

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n).$$

B-495 *Proposed by Philip L. Mana, Albuquerque, NM*

Characterize an infinite sequence whose first 24 terms are given in the following:

1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55,
61, 64, 77, 81, 85, 86, 90, 91, ...

[Note that all perfect squares occur in the sequence.]

SOLUTIONS

Squares and Products of Consecutive Integers

B-466 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A_n = 1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \cdots + (-1)^{n-1}n(n+1)$.

(a) Determine the values of n for which $2A_n$ is a perfect square.

(b) Determine the value of n for which $|A_n|/2$ is the product of two consecutive positive integers.

Solution by Graham Lord, Québec, Canada

$A_1 = 2$, $A_2 = -4$, $A_3 = 8$, $A_4 = -12$, and one can easily establish (by induction)

$$A_{2m-1} = 2m^2 \quad \text{and} \quad A_{2m} = -2m(m+1).$$

Then $2A_n$ is a perfect square if n is odd and $|A_n|/2$ is the product of two consecutive positive integers if n is even. But since the equation

$$x^2 = y^2 + 1$$

has no solution in positive integers, $2|A_n|$ cannot be a perfect square when n is even and $|A_n|/2$ cannot be the product of two consecutive integers when n is odd.

Also solved by Paul S. Bruckman, H. Klauser, P. V. Satyanarayana Murty, Bob Prielipp, Sahib Singh, Gregory Wulczyn, a solver at the Madchengymnasium Essen-Borbeck, and the proposer.

A's into B's

B-467 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A_n be as in B-466 and let

$$B_n = \sum_{i=1}^{n} \sum_{k=1}^{i} k.$$

For which positive integers n is $|A_n|$ an integral divisor of B_n?

Solution by Graham Lord, Québec, Canada

Note that $2 = A_1$ does not divide $B_1 = 1$. As $B_n = n(n + 1)(n + 2)/6$, then

$$B_{2m-1} = m(4m^2 - 1)/3,$$

which is evidently not divisible by $A_{2m-1} = 2m^2$, for $m > 1$. And for n even,

$$B_{2m} = 2m(2m + 1)/3,$$

which will be divisible by $|A_{2m}| = 2m(m + 1)$ as long as $(2m + 1)/3$ is an integer; that is, if $m \equiv 1 \pmod 3$ or, equivalently, $n \equiv 2 \pmod 6$.

Also solved by Paul S. Bruckman, H. Klauser, P. V. Satyanarayana Murty, Bob Prielipp, Sahib Singh, the solver at the Madchengymnasium Essen-Borbeck, and the proposer.

Fibonacci Sines

B-468 *Proposed by Miha'ly Bencze, Brasov, Romania*

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

Solution by Sahib Singh, Clarion State College, Clarion, PA

Let $a_1 = \text{Sin } A$, $a_2 = \text{Sin } B$, where A, B are in radian measure and belong to the open interval $(0, \pi/2)$. Thus

$$a_3 = \text{Sin}(A + B),\ a_4 = \text{Sin}(A + 2B),$$

and in general, by induction technique, we conclude that

$$a_n = \text{Sin}(AF_{n-2} + BF_{n-1}), \text{ where } n \geq 1.$$

Also solved by Paul S. Bruckman, L. Carlitz, and the proposer.

Base F_n Expansions

B-469 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

Describe the appearance in base F_n notation of:

(a) $1/F_{n-1}$ for $n \geq 5$; (b) $1/F_{n+1}$ for $n \geq 3$.

Solution by Graham Lord, Québec, Canada

Let $F_{2n-2} = u$, $F_{2n-1} - 1 = v$, $F_{2n} - 2 = w$, $F_{2n} - 1 = x$, $F_{2n} = y$, and $F_{2n+1} = z$. The identity $F_{m-1}F_{m+1} - F_m^2 = (-1)^m$ gives, for $m = 2n + 1$:

$$1/F_{2n} = F_{2n+2}/(z^2 - 1) = (z + y)(z^{-2} + z^{-4} + z^{-6} + \cdots)$$
$$= z^{-1} + yz^{-2} + z^{-3} + yz^{-4} + \cdots,$$

which is $.\overline{1y}$ in base F_{2n+1}. And

$$1/F_{2n+2} = F_{2n}/(z^2 - 1),$$

which is $.\overline{0y}$ in base F_{2n+1}. The same identity for $m = 2n$ yields:

$$1/F_{2n-1} = F_{2n+1}/(y^2 + 1) = (F_{2n} + F_{2n-1})(F_{2n}^2 - 1)/(y^4 - 1)$$
$$= [F_{2n}^3 + F_{2n}^2(F_{2n-1} - 1) + F_{2n}(F_{2n} - 2) + F_{2n-2}]/(y^4 - 1),$$

which is $.\overline{1vwu}$ in base F_{2n}. Similarly,

$$1/F_{2n+1} = F_{2n-1}(F_{2n}^2 - 1)/(y^4 - 1)$$
$$= [F_{2n}^2(F_{2n-1} - 1) + F_{2n}(F_{2n} - 1) + F_{2n-2}]/(y^4 - 1),$$

which is $.\overline{0vxu}$ in base F_{2n}. The lower bounds imposed on the subscripts guarantee the digits are nonnegative.

Also solved by Paul S. Bruckman, L. Carlitz, Bob Prielipp, J. O. Shallit, Sahib Singh, and the proposer.

3 Term A.P.

B-470 *Proposed by Larry Taylor, Rego Park, NY*

Find positive integers a, b, c, r, and s, and choose each of G_n, H_n, and I_n to be F_n or L_n, so that

$$aG_n,\ bH_{n+r},\ \text{and}\ cI_{n+s}$$

are in arithmetic progression for $n \geqslant 0$ and this progression is 6, 6, 6 for some n.

Solution by Paul S. Bruckman, Carmichael, CA

In order for the indicated quantities to equal 6, they must lie in the set

$$S = \{6F_1,\ 6F_2,\ 6L_1,\ 3F_3,\ 3L_0,\ 2F_4,\ 2L_2\}$$

for some n. This means that for all n, the indicated quantities must lie in the set T_n, defined as follows:

$$\{6F_n,\ 6F_{n+1},\ 6F_{n+2},\ 6L_n,\ 6L_{n+1},\ 3F_n,\ 3F_{n+1},\ 3F_{n+2},\ 3F_{n+3},\ 3L_n,$$
$$2F_n,\ 2F_{n+1},\ 2F_{n+2},\ 2F_{n+3},\ 2F_{n+4},\ 2L_n,\ 2L_{n+1},\ 2L_{n+2}\}.$$

Of the 18 elements of T_n, 3 are to be in arithmetic progression for all n. We may choose n sufficiently large so that no duplication of elements occurs in T_n, e.g., $n = 5$. Thus,

$$T_5 = \{10, 15, 16, 22, 24, 26, 30, 33, 36, 39, 42, 48, 58, 63, 66, 68, 78, 108\}.$$

Considering all possible combinations, we find that the only triplets which are subsets of T_5 in arithmetic progression are as follows:

(10,16,22), (10,26,42), (10,39,68), (15,24,33), (15,39,63), (16,26,36),
(16,42,68), (22,24,26), (22,26,30), (24,30,36), (24,33,42), (24,36,48),
(24,66,108), (26,42,58), (30,33,36), (30,36,42), (30,39,48), (30,48,66),
(33,36,39), (33,48,63), (36,39,42), (36,42,48), (48,63,78), (48,78,108),
(58,63,68), and (58,68,78).

We then relate each triplet above to the appropriate multiple of a Fibonacci or Lucas number, e.g., $(10,16,22) = (2F_5,\ 2F_6,\ 2L_5)$. From the resulting set of 26 triplets, we exclude those where the *smallest* subscript is repeated (which is a consequence of the requirement that r and s be positive); thus, we would not count (10,16,22), since the subscript 5 is repeated. We thus reduce the foregoing set of triplets to the following set:

$(2F_5,\ 2F_7,\ 2F_8)$, $(2F_5,\ 3F_7,\ 2F_9)$, $(3F_5,\ 3F_7,\ 3F_8)$, $(2F_6,\ 2F_8,\ 2F_9)$,
$(2L_5,\ 3F_6,\ 2F_7)$, $(3F_6,\ 6F_5,\ 2L_6)$, $(3F_6,\ 3L_5,\ 2F_8)$, $(3F_6,\ 6L_5,\ 6L_6)$,
$(6F_5,\ 2L_6,\ 2F_8)$, $(6F_5,\ 3F_7,\ 6F_6)$, $(3L_5,\ 2L_6,\ 3F_7)$, $(3L_5,\ 6F_6,\ 3F_8)$,
$(2L_6,\ 3F_7,\ 2F_8)$, $(6F_6,\ 3F_8,\ 7F_7)$, and $(2L_7,\ 3F_8,\ 2F_9)$.

Each of the foregoing triplets corresponds in the general case to a triplet which is a subset of T_n; if we form these corresponding triplets, however, with the smallest subscript in the triplets from T_5 replaced by n, we obtain some triplets which must be rejected, since they do not reduce to (6, 6, 6) for any value of n. To illustrate, the triplet ($2F_5$, $2F_7$, $2F_8$) suggests the possible triplet ($2F_n$, $2F_{n+2}$, $2F_{n+3}$) in the general case; however, the latter triplet clearly can never equal (6, 6, 6) for any n. This further restriction reduces the total set of possible triplets to four possibilities, and these turn out to be acceptable solutions:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3}),\ (6F_n,\ 3F_{n+2},\ 6F_{n+1}),$$
$$(2L_n,\ 3F_{n+1},\ 2F_{n+2}),\ (3L_n,\ 6F_{n+1},\ 3F_{n+3}).$$

The above triplets assume the values (6, 6, 6) for n = 1, 1, 2, and 0, respectively. It is an easy exercise to verify that the above triplets are in arithmetic progression for all n, and the proof is omitted here.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

4 Term A.P.

B-471 *Proposed by Larry Taylor, Rego Park, NY*

Do there exist positive integers d and t such that

$$aG_n,\ bH_{n+r},\ cI_{n+s},\ dJ_{n+t}$$

are in arithmetic progression, with J_n equal to F_n or L_n and everything else as in B-470?

Solution by Paul S. Bruckman, Carmichael, CA

Any quadruplet consisting of the indicated quantities must contain a solution of B-470 as its first three elements. Referring to that solution, if we set $n = 5$, for example, we obtain the triplets:

$$(30,36,42),\ (30,39,48),\ (22,24,26),\text{ and } (33,48,63).$$

Therefore, any solution of this problem must reduce, for $n = 5$, to the quadruplets:

$$(30,36,42,48),\ (30,39,48,57),\ (22,24,26,28),\text{ or } (33,48,63,78).$$

Each element of any quadruplet must be of the form kU_m, where k = 2, 3, or 6, U is F or L, and m is a nonnegative integer. However, 57 and 28 are not of this form ($57 = 3 \cdot 19$, and 19 is neither a Fibonacci nor a Lucas number; $28 = 2 \cdot 14$, and 14 is neither a Fibonacci nor a Lucas number). We must therefore eliminate the second and third of the above indicated quadruplets. This leaves the following two triplets as possibly generating acceptable solutions of this problem:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3}) \quad \text{or} \quad (3L_n,\ 6F_{n+1},\ 3F_{n+3}).$$

If these *do* generate acceptable solutions to this problem, the fourth element of the desired quadruplet must equal twice the third element, less the second element. Thus, if x_i denotes the missing fourth element corresponding to the ith triplet above ($i = 1$ or 2), then

$$x_1 = 4F_{n+3} - 2L_{n+1} = 4F_{n+2} + 4F_{n+1} - 2F_{n+2} - 2F_n$$

$$= 2F_{n+2} - 2F_n + 4F_{n+1} = 6F_{n+1};$$

also,

$$x_2 = 6F_{n+3} - 6F_{n+1} = 6F_{n+2}.$$

This suggests the possible solutions:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3},\ 6F_{n+1}) \quad \text{and} \quad (3L_n,\ 6F_{n+1},\ 3F_{n+3},\ 6F_{n+2}).$$

It only remains to verify that these quadruplets assume the values (6, 6, 6, 6) for the same values of n which generated the triplets (6, 6, 6) in B-470, i.e., for $n = 1$ and $n = 0$, respectively. Obviously, this is the case. Therefore, the above two solutions are the only solutions to this problem.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

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