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HARMONIC SUMS AND THE ZETA FUNCTION

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1. SUMMARY

Consider the harmonic sequence

$$H_n = \sum_{k=1}^{n} k^{-1}, \; n \geqslant 1,$$

and the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}, \; \mathrm{Re}(s) > 1.$$

Recently, Bruckman [2] proposed the problem of showing

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3).$$

See also Klamkin [3] and Steinberg [4]. Presently, we establish the following generalization.

<u>Theorem</u>

Let H_n and $\zeta(s)$ be as above. Then

$$\text{(i)} \quad \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+1}} = \frac{1}{2} \sum_{j=2}^{2n} (-1)^j \zeta(j)\zeta(2n+2-j), \; n \geqslant 1,$$

and

$$\text{(ii)} \quad \sum_{k=1}^{\infty} \frac{H_k}{k^n} = \left(1 + \frac{n}{2}\right)\zeta(n+1) - \frac{1}{2} \sum_{j=2}^{n-1} \zeta(j)\zeta(n+1-j), \; n \geqslant 2.$$

Here and in the sequel, as usual,

$$\sum_{j=j_0}^{n} c_j = 0 \text{ if } n < j_0.$$

The series which will be manipulated are readily shown to be absolutely convergent, so that summation signs may be reversed.

The proof of the theorem will be given in Section 2 after some auxiliary results have been derived. Some further generalizations are given in Section 3, and an open problem is stated.

2. AUXILIARY RESULTS AND PROOF OF THE THEOREM

Define the generalized harmonic sequence

$$H_0^{(m)} = 0 \text{ and } H_n^{(m)} = \sum_{\ell=1}^{n} \ell^{-m},\ m \geqslant 1,\ n \geqslant 1, \tag{2.1}$$

and set

$$\overline{H}_n^{(1)} = \gamma - H_n^{(1)} \quad \text{and} \quad \overline{H}_n^{(m)} = \zeta(m) - H_n^{(m)},\ m \geqslant 2,\ n \geqslant 0, \tag{2.2}$$

where γ is Euler's constant. Note that

$$\sum_{\ell=1}^{N} \frac{n}{\ell(\ell+n)} = \sum_{\ell=1}^{N}\left(\frac{1}{\ell} - \frac{1}{\ell+n}\right) = H_N - \sum_{\ell=n+1}^{N+n}\frac{1}{\ell} = H_N - H_{N+n} + H_n$$

$$= H_n + (H_N - \log N) - [H_{N+n} - \log(N+n)] - \log\left(1 + \frac{n}{N}\right);$$

therefore, using the well-known limiting expression

$$\lim_{N\to\infty}(H_N - \log N) = \gamma, \tag{2.2a}$$

it follows that

$$H_n = H_n^{(1)} = \sum_{\ell=1}^{\infty} \frac{n}{\ell(\ell+n)},\ n \geqslant 0; \tag{2.3}$$

it also follows from (2.1) and (2.2) that

$$\overline{H}_n^{(m)} = \sum_{\ell=1}^{\infty} (\ell+n)^{-m},\ m \geqslant 2,\ n \geqslant 0. \tag{2.4}$$

Now define the sums

$$S_n^{(m)} = \sum_{k=1}^{\infty} \frac{H_k^{(m)}}{k^n},\ m \geqslant 1,\ n \geqslant 2, \tag{2.5}$$

and

$$\overline{S}_n^{(m)} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^{(m)}}{k^n},\ m \geqslant 2,\ n \geqslant 1, \tag{2.6}$$

which may be shown to exist. $S_n^{(1)}$ exists because $H_k = O(\log k)$ and

$$\sum_{k=1}^{\infty} \frac{\log k}{k^n}$$

exists for all $n \geqslant 2$. Also

$$\overline{S}_1^{(m)} = S_m^{(1)} - \zeta(m+1),$$

as will be shown in Lemma 2.1, so $\overline{S}_1^{(m)}$ exists for all $m \geqslant 2$. These sums are related to the zeta function as follows.

Lemma 2.1

Let $S_m^{(n)}$ and $\bar{S}_n^{(m)}$ be as in (2.5) and (2.6), respectively, and let $\zeta(\cdot)$ be the Riemann zeta function. Then

(i) $S_m^{(n)} = \bar{S}_n^{(m)} + \zeta(m+n),\ m \geqslant 2,\ n \geqslant 1,$

and

(ii) $S_m^{(n)} + S_n^{(m)} = \zeta(m+n) + \zeta(m)\zeta(n),\ m \geqslant 2,\ n \geqslant 2.$

Proof: (i) Clearly,

$$S_m^{(n)} = \sum_{k=1}^{\infty} \frac{H_k^{(n)}}{k^m} = \sum_{k=1}^{\infty} \frac{1}{k^{m+n}} + \sum_{k=1}^{\infty} \frac{H_{k-1}^{(n)}}{k^m}$$

$$= \zeta(m+n) + \sum_{k=1}^{\infty} \frac{H_k^{(n)}}{(k+1)^m}, \text{ by (2.5) and (2.1).}$$

Next,

$$\sum_{k=1}^{\infty} \frac{H_k^{(n)}}{(k+1)^m} = \sum_{k=1}^{\infty} (k+1)^{-m} \sum_{\ell=1}^{k} \ell^{-n}, \text{ by (2.1),}$$

$$= \sum_{\ell=1}^{\infty} \ell^{-n} \sum_{k=\ell}^{\infty} (k+1)^{-m}$$

$$= \sum_{\ell=1}^{\infty} \ell^{-n} \sum_{k=1}^{\infty} (k+\ell)^{-m}$$

$$= \bar{S}_n^{(m)}, \text{ by (2.4) and (2.6).}$$

The last two relations establish (i).

(ii) Relation (2.6) gives

$$\bar{S}_n^{(m)} = \zeta(m)\zeta(n) - S_n^{(m)},\ m \geqslant 2,\ n \geqslant 2,$$

by means of (2.2) and (2.5). This along with (i) establishes (ii).

Lemma 2.2

For each integer $m_1, m_2 \geqslant 1$, and $n_1 \neq n_2 \geqslant 0$, set

$$A_{1j} = A_{1j}(m_1, m_2, n_1, n_2)$$

$$= (-1)^{m_1+j} \binom{m_1 + m_2 - 1 - j}{m_2 - 1} (n_2 - n_1)^{-m_1-m_2+j},$$

and

$$A_{2j} = A_{2j}(m_1, m_2, n_1, n_2) = (-1)^{m_2+j}\binom{m_1 + m_2 - 1 - j}{m_1 - 1}(n_1 - n_2)^{-m_1-m_2+j},$$

and let $\overline{H}_{n_1}^{(j)}$ and $\overline{H}_{n_2}^{(j)}$ be given by (2.2). Then

$$\sum_{k=1}^{\infty} \frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = \sum_{i=1}^{2}\sum_{j=1}^{m_i} A_{ij}\overline{H}_{n_i}^{(j)}.$$

Proof: Expanding $(k + n_1)^{-m_1}(k + n_2)^{-m_2}$ into partial fractions, we obtain (by residue theory or otherwise)

$$(k + n_1)^{-m_1}(k + n_2)^{-m_2} = \sum_{j=1}^{m_1} \frac{A_{1j}}{(k+n_1)^j} + \sum_{j=1}^{m_2} \frac{A_{2j}}{(k+n_2)^j} \tag{2.7}$$

with A_{1j} and A_{2j} as defined above. We see that $A_{21} = -A_{11}$. Then, summing in (2.7) over $k \geqslant 1$, and using (2.2) and (2.4), we obtain

$$\sum_{k=1}^{\infty} \frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = \sum_{k=1}^{\infty}\left\{\sum_{j=1}^{m_1} \frac{A_{1j}}{(k+n_1)^j} + \sum_{j=1}^{m_2} \frac{A_{2j}}{(k+n_2)^j}\right\}$$

$$= \sum_{k=1}^{\infty}\left\{\left(\frac{A_{11}}{k+n_1} + \frac{A_{21}}{k+n_2}\right) + \sum_{j=2}^{m_1} \frac{A_{1j}}{(k+n_1)^j} + \sum_{j=2}^{m_2} \frac{A_{2j}}{(k+n_2)^j}\right\};$$

now

$$\sum_{k=1}^{\infty}\left(\frac{A_{11}}{k+n_1} + \frac{A_{21}}{k+n_2}\right) = A_{11}\sum_{k=1}^{\infty}\left(\frac{1}{k+n_1} - \frac{1}{k+n_2}\right)$$

$$= A_{11}\sum_{k=1+n_1}^{n_2} \frac{1}{k} \quad (\text{if } n_1 < n_2)$$

$$= A_{11}(H_{n_2} - H_{n_1}) = A_{11}(\overline{H}_{n_1}^{(1)} - \overline{H}_{n_2}^{(1)})$$

$$= A_{11}\overline{H}_{n_1}^{(1)} + A_{21}\overline{H}_{n_2}^{(1)}.$$

A similar conclusion follows if $n_1 \geqslant n_2$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = A_{11}\overline{H}_{n_1}^{(1)} + A_{21}\overline{H}_{n_2}^{(1)} + \sum_{j=2}^{m_1} A_{1j}\overline{H}_{n_1}^{(j)} + \sum_{j=2}^{m_2} A_{2j}\overline{H}_{n_2}^{(j)}$$

(continued)

$$= \sum_{i=1}^{2} \sum_{j=1}^{m_i} A_{ij} \overline{H}_{n_i}^{(j)},$$

which was to be shown.

Lemma 2.2 will be utilized to establish the following:

Lemma 2.3

Let $S_n^{(m)}$ and $\overline{S}_m^{(n)}$ be given by (2.5) and (2.6), respectively. Then

$$(-1)^{m+1}\overline{S}_m^{(n)} = \binom{m+n-2}{n-1} S_{m+n-1}^{(1)} - \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \overline{S}_{m+n-j}^{(j)}$$
$$- \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j)\zeta(m+n-j),$$
$$m \geqslant 1,\ n \geqslant 2.$$

Proof: We have

$$\overline{S}_m^{(n)} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^{(n)}}{k^m} = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^m (k+\ell)^n}, \text{ by (2.4) and (2.6)},$$

$$= \sum_{\ell=1}^{\infty} \left\{ \sum_{j=1}^{m} A_{1j} \overline{H}_0^{(j)} + \sum_{j=1}^{n} A_{2j} \overline{H}_\ell^{(j)} \right\}, \text{ by Lemma 2,2},$$

$$= \sum_{\ell=1}^{\infty} \left\{ A_{11} H_\ell^{(1)} + \sum_{j=2}^{m} A_{1j} \zeta(j) + \sum_{j=2}^{n} A_{2j} \overline{H}_\ell^{(j)} \right\}, \text{ by (2.1), (2.2) and } A_{21} = -A_{11},$$

$$= (-1)^{m+1} \binom{m+n-2}{n-1} \sum_{\ell=1}^{\infty} \frac{H_\ell^{(1)}}{\ell^{m+n-1}}$$
$$+ (-1)^m \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j) \sum_{\ell=1}^{\infty} \frac{1}{\ell^{m+n-j}}$$
$$+ (-1)^m \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \sum_{\ell=1}^{\infty} \frac{\overline{H}_\ell^{(j)}}{\ell^{m+n-j}}$$

$$= (-1)^{m+1} \binom{m+n-2}{n-1} S_{m+n-1}^{(1)}$$
$$+ (-1)^m \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j)\zeta(m+n-j)$$

(continued)

$$+ (-1)^m \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \overline{S}^{(j)}_{m+n-j}, \text{ by (2.5) and (2.6)},$$

from which the lemma follows.

Proof of the Theorem

(i) Utilizing (2.3) and Lemma 2.2 with $m_1 = 2n$, $m_2 = 1$, $n_1 = 0$, and $n_2 = \ell$, we get

$$\sum_{k=1}^{\infty} \frac{H_k}{k^{2n+1}} = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{\ell=1}^{\infty} \frac{k}{\ell(k+\ell)} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{k=1}^{\infty} \frac{1}{k^{2n}(k+\ell)}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left\{ \sum_{j=1}^{2n} A_{1j} \overline{H}_0^{(j)} + A_{21} \overline{H}_\ell^{(1)} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left\{ \left(-\frac{\overline{H}_0^{(1)}}{\ell^{2n}} + \frac{\overline{H}_\ell^{(1)}}{\ell^{2n}} \right) + \sum_{j=2}^{2n} (-1)^j \frac{\overline{H}_0^{(j)}}{\ell^{2n+1-j}} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{-H_\ell^{(1)}}{\ell^{2n+1}} + \sum_{j=2}^{2n} (-1)^j \zeta(j) \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2n+2-j}}, \text{ by (2.1) and (2.2)},$$

$$= -\sum_{\ell=1}^{\infty} \frac{H_\ell^{(1)}}{\ell^{2n+1}} + \sum_{j=2}^{2n} (-1)^j \zeta(j)\zeta(2n+2-j),$$

from which (i) follows.

(ii) Setting $m = 1$ in Lemma 2.3, we get

$$\overline{S}_1^{(n)} = S_n^{(1)} - \sum_{j=2}^{n} \overline{S}^{(j)}_{n+1-j}, \quad n \geqslant 2,$$

and from Lemma 2.1(i) we have

$$\overline{S}^{(j)}_{n+1-j} = S_j^{(n+1-j)} - \zeta(n+1), \quad j \geqslant 2, \ n \geqslant 2.$$

In particular,

$$\overline{S}_1^{(n)} = S_n^{(1)} - \zeta(n+1), \quad n \geqslant 2.$$

It follows that

$$\zeta(n+1) = \sum_{j=2}^{n} \overline{S}^{(j)}_{n+1-j} = \sum_{j=2}^{n} \left\{ S_j^{(n+1-j)} - \zeta(n+1) \right\}, \quad n \geqslant 2,$$

or, equivalently,

$$S_n^{(1)} = n\zeta(n+1) - \sum_{j=2}^{n-1} S_j^{(n+1-j)}, \quad n \geqslant 2. \tag{2.8}$$

Next, Lemma 2.1(ii) gives

$$S_j^{(n+1-j)} + S_{n+1-j}^{(j)} = \zeta(n+1) + \zeta(j)\zeta(n+1-j), \ j \geq 2, \ n \geq 3,$$

so that (by a change in variable from j to $n+1-j$)

$$\begin{aligned} 2\sum_{j=2}^{n-1} S_j^{(n+1-j)} &= \sum_{j=2}^{n-1}\left\{S_j^{(n+1-j)} + S_{n+1-j}^{(j)}\right\} \\ &= (n-2)\zeta(n+1) + \sum_{j=2}^{n-1}\zeta(j)\zeta(n+1-j), \ n \geq 2. \end{aligned} \tag{2.9}$$

Relations (2.8) and (2.9), along with (2.1) and (2.5), establish (ii).

As a byproduct of the theorem, we get the following interesting result, if we replace n by $2n+1$ in (ii) of the theorem, eliminate the series, then replace $n+1$ by n.

Corollary

$$\zeta(2n) = \frac{2}{2n+1}\sum_{j=1}^{n-1}\zeta(2j)\zeta(2n-2j), \ n \geq 2.$$

Remark: Taking into account that

$$\zeta(2n) = (-1)^{n-1}2^{2n-1}\pi^{2n}[(2n)!]^{-1}B_{2n}, \ n \geq 1,$$

from [1], where B_n are the Bernoulli numbers, the above relation becomes

$$B_{2n} = -\frac{1}{2n+1}\sum_{j=1}^{n-1}\binom{2n}{2j}B_{2j}B_{2n-2j}, \ n \geq 2.$$

3. FURTHER GENERALIZATIONS

In this section, we give the following additional results, which express generalized harmonic sums in terms of the zeta function.

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^{2n+1}} &= \zeta(2)\zeta(2n+1) - \frac{(n+2)(2n+1)}{2}\zeta(2n+3) \\ &\quad + 2\sum_{j=2}^{n+1}(j-1)\zeta(2j-1)\zeta(2n+4-2j), \ n \geq 1. \end{aligned} \tag{3.1}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k^n} = \frac{1}{2}[\zeta(2n) + \zeta(n)\zeta(n)], \ n \geq 2. \tag{3.2}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4} = -\frac{1}{3}\zeta(6) + \zeta(3)\zeta(3). \tag{3.3a}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^4} = 18\zeta(7) - 10\zeta(2)\zeta(5). \tag{3.3b}$$

Relation (3.1) follows from Lemma 2.3 (by setting $n = 2$ and replacing m by $2m + 1$), Lemma 2.1, and part (ii) of the theorem. Relation (3.2) follows immediately from Lemma 2.1(ii) by setting $m = n$. Finally, relations (3.3a) and (3.3b) can be derived from Lemma 2.3 by setting the appropriate values of m and n. We also note that the sum

$$\sum_{k=1}^{\infty} \frac{H_k^{(2\ell+1-n)}}{k^n} \quad \left(n \geqslant 5,\ \ell \geqslant \left[\frac{n+1}{2}\right]\right)$$

may be obtained from Lemma 2.3 by means of some algebra that becomes progressively cumbersome with increasing n.

It is still an open question to give a closed form of

$$\sum_{k=1}^{\infty} \frac{H_k^{(m)}}{k^n}$$

for any integers $m \geqslant 1$ and $n \geqslant 2$ in terms of the zeta function.

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