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A PROPERTY OF FIBONACCI AND TRIBONACCI NUMBERS

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1. INTRODUCTION

The Fibonacci numbers are defined by setting

$$a_1 = a_2 = 1 \text{ and } a_{n+1} = a_n + a_{n-1} \text{ for } n \geqslant 2.$$

A related family of sequences are the t-bonacci numbers (where $t \geqslant 2$ is an integer). These are defined by setting

$$a_1 = 1,\ a_n = 2^{n-2} \text{ for } 2 \leqslant n \leqslant t$$

and

$$a_{n+1} = a_n + \cdots + a_{n-t+1} \text{ for } n \geqslant t.$$

Thus, for $t = 2$ we obtain the Fibonacci numbers again, and for $t = 3$ we obtain the so-called Tribonacci numbers.

The Fibonacci numbers have many interesting properties. The property of interest to us here is that this sequence satisfies the equation

$$\Delta a_n = a_{n-1} \qquad (n \geqslant 2),$$

where Δ denotes the forward difference operator. The Tribonacci numbers satisfy

$$\Delta^3 a_n = 2a_{n-2} \qquad (n \geqslant 3).$$

We call a sequence (a_n) that satisfies an equation of the form

$$\Delta^k a_n = ma_{n-r} \qquad (n > r), \tag{1}$$

a *self-generating sequence with parameters* (k, m, r). We abbreviate this to SGS(k, m, r). [We will work under the convenient assumption that k, m, and r are integers and that $k \geqslant 1$. Similarly, our sequences (a_n) will be integral.]

Thus, the Fibonacci numbers are an SGS(1, 1, 1) and the Tribonacci numbers form an SGS(3, 2, 2). This immediately suggests the question of whether, for any $t \geqslant 4$, the t-bonacci numbers form a self-generating sequence. The main result of this paper is as follows.

Theorem 1

The Fibonacci sequence is an SGS(1, 1, 1); the Tribonacci sequence is an SGS(3, 2, 2). For $t \geqslant 4$, no t-bonacci sequence is self-generating.

2. PROOF OF THEOREM 1

Let

$$F(x) = \sum_{n=1}^{\infty} a_n x^n$$

denote the generating function (G.F.) of the sequence (a_n) and let $\Delta^j F(x)$ denote the G.F. of the sequence of jth forward differences $(\Delta^j a_n)$.

Lemma 1

For $j \geqslant 1$, we have

$$\Delta^j F(x) = \frac{1}{x^j}[(1 - x)^j F(x) - xp_{j-1}(x)], \tag{2}$$

where $p_{j-1}(x)$ denotes a polynomial of degree at most $j - 1$.

Lemma 1 can be proved by induction on j. We leave the details as an exercise.

Now let (a_n) be an SGS(k, m, r). In order to satisfy (1), we have to subtract from $\Delta^k F(x)$ its first r terms [i.e., a polynomial $q_r(x)$ of degree at most r] and equate the rest with $mx^r F(x)$:

$$\frac{1}{x^k}[(1 - x)^k F(x) - xp_{k-1}(x)] - q_r(x) = mx^r F(x).$$

From this equation, we immediately obtain:

Theorem 2

The generating function of an SGS(k, m, r) is of the form

$$F(x) = \frac{p_{k+r}(x)}{(1 - x)^k - mx^{k+r}}, \tag{3}$$

where $p_{k+r}(x)$ is a polynomial of degree at most $k + r$ with zero constant term. □

Remark 1: It can be shown that any sequence with generating function of the form given in (3) is an SGS(k, m, r). We will not prove this because we will not make use of it here.

The G.F. for the t-bonacci sequence is

$$F(x) = \frac{x}{1 - x - \cdots - x^t};$$

hence, a necessary condition for this sequence to be self-generating is that the zeros $\alpha, \beta, \gamma, \ldots$ of $1 - x - \cdots - x^t$ are also zeros of the polynomial $(1 - x)^k - mx^{k+r}$ appearing in the form of $F(x)$ given in Theorem 1. Our aim is to show that for $t \geqslant 4$ we can find three zeros α, β, γ for which this necessary condition is violated. Thus, it will be useful to list some facts about the roots of

$$1 - x - \cdots - x^t = 0. \tag{4}$$

Remark 2: We observe that no root of (4) equals 1. Now, multiplying (4) by $1 - x$ and collecting terms transforms (4) into

$$x^{t+1} - 2x + 1 = 0. \tag{5}$$

Remark 3: A geometrical argument about the curves $y = x^{t+1}$ and $y = 2x - 1$ shows that for odd t there is exactly one, for even t there are exactly two, real roots of (5) not equal to 1. For all t, one of these tends monotonically to -1 from the left as t increases. In [1], the positive real roots have been calculated. For $t = 6$ this root is $\alpha = 0.504138\ldots$; hence, for $t \geqslant 6$ we have $\alpha < 0.505$.

Remark 4: In [2], it was proved that (5) has exactly one root z with $|z| < 1$ and one with $|z| = 1$; all other roots satisfy $|z| > 1$. We shall now give an upper bound for the absolute values of these roots.

Lemma 2

The roots of (5) with $|z| > 1$ satisfy $|z| < \quad 3$.

Proof: Let z be a root of (5) with $|z| > 1$. Then, since

$$|z||z^t - 2| = |-1| = 1,$$

we have $|z^t - 2| < 1$, which implies $|z^t| < 3$ and, therefore, $|z| < \sqrt[t]{3}$. □

Combined with the previous lemma, our next result approximately determines the positions of the roots of (5).

Lemma 3

For each j with $1 \leqslant j \leqslant \frac{t-1}{2}$, Eq. (5) has a root z_j with

$$\arg z_j \in I_j = \left(\frac{2j\pi}{t}, \frac{2j\pi}{t-1}\right].$$

Proof: We use Gauss's method for trinomial equations (see e.g. [3], pp. 397- 398). Write $z = \rho(\cos\varphi + i\sin\varphi)$. Then, if $z^{t+1} - 2z + 1$ is zero, we must have

$$\rho^{t+1}\cos(t+1)\varphi - 2\rho\cos\varphi + 1 = 0; \tag{6}$$

$$\rho^{t+1}\sin(t+1)\varphi - 2\rho\sin\varphi = 0. \tag{7}$$

From (7), we get

$$\rho^t = \frac{2\sin\varphi}{\sin(t+1)\varphi}.$$

Substituting this into (6) and using the trigonometric addition formulas, we obtain

$$\rho = \frac{\sin(t+1)\varphi}{2\sin t\varphi}. \tag{8}$$

Upon substitution into (7), this yields

$$2^{t+1}\sin^t t\varphi\sin\varphi - \sin^{t+1}(t+1)\varphi = 0, \tag{9}$$

which determines φ. Denote the left-hand side of (9) by $f(\varphi)$. Then

$$f\left(\frac{2j\pi}{t}\right) < 0$$

whereas

$$f\left(\frac{2j\pi}{t-1}\right) \geqslant 0.$$

By the continuity of f, the lemma follows. □

Now let $t \geqslant 4$ and let α, β, and γ denote three nonconjugate distinct roots of (4). If the t-bonacci sequence was self-generating, we would have $(1-\alpha)^k = m\alpha^{k+r}$ as well as $(1-\beta)^k = m\beta^{k+r}$ for some k, m, and r; hence,

$$\left(\frac{1-\alpha}{1-\beta}\right)^k = \left(\frac{\alpha}{\beta}\right)^{k+r}.$$

An analogous equation holds for α and γ. Taking logarithms, we get

$$k\log\frac{1-\alpha}{1-\beta} - (k+r)\log\frac{\alpha}{\beta} = 0$$

and

$$k\log\frac{1-\alpha}{1-\gamma} - (k+r)\log\frac{\alpha}{\gamma} = 0.$$

To obtain nontrivial solutions for given k and r, the two equations must be linearly dependent. Therefore, considering the absolute values, we must have

$$\log\left|\frac{1-\alpha}{1-\beta}\right| \log\left|\frac{\alpha}{\gamma}\right| = \log\left|\frac{1-\alpha}{1-\gamma}\right| \log\left|\frac{\alpha}{\beta}\right|. \tag{10}$$

Denote the left- and right-hand sides of (10) by L and R, respectively. Our aim is to find roots α, β, and γ for which $L \neq R$.

Let t be even, $t \geqslant 6$. Take as α the positive real root, as β the negative real root, and as γ a root with $0 < \arg\gamma < 2\pi/5$. (Such a exists, by Lemma 3.) Then the following inequalities hold, by virtue of Remark 3 and Lemma 2:

$$\begin{array}{rr} 0.5 < |\alpha| < 0.505 & 0.495 < |1-\alpha| < 0.5 \\ |\beta| < 1.201 & 2 < |1-\beta| \\ 1 < |\gamma| & |1-\gamma| < 1.304 \end{array}$$

From these, we calculate $L > 0.947$ and $R < 0.849$.

Now let t be odd, $t \geqslant 7$. As α we take the positive real root, as β a root with $6\pi/7 < \arg\beta < \pi$, and as γ a root with $0 < \arg\gamma < 2\pi/6$. The resulting inequalities are

$$\begin{array}{rr} 0.5 < |\alpha| < 0.505 & 0.495 < |1-\alpha| < 0.5 \\ |\beta| < 1.17 & 1.94 < |1-\beta| \\ 1 < |\gamma| & |1-\gamma| < 1.094 \end{array}$$

and we obtain $L > 0.926$ and $R < 0.675$.

The remaining cases, $t = 4$ and $t = 5$, can be settled by approximate calculation of φ and ρ using (8) and (9); again, roots can be found for which $L \neq R$. The details will be omitted here. □

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