◆◇◆◇◆

PROPERTIES OF POLYNOMIALS HAVING FIBONACCI NUMBERS FOR COEFFICIENTS

D. H. LEHMER and EMMA LEHMER
University of California, Berkeley, CA 94720
(Submitted January 1982)

In memory of Vern Hoggatt, Jr.

It is unusual when one comes across a sequence of polynomials whose coefficients, roots, and sums of powers can all be given explicitly. It is our purpose to expose such a sequence of polynomials involving Fibonacci numbers.

The general polynomial in question is of even degree, which it will be convenient to take as $2n - 2$. The coefficients are the first n Fibonacci numbers as follows:

$$P_n(x) = x^{2n-2} + x^{2n-3} + 2x^{2n-4} + \cdots + F_n x^{n-1} - F_{n-1}x^{n-2} + F_{n-2}x^{n-3}$$
$$- F_{n-3}x^{n-4} + \cdots + (-1)^n x - (-1)^n.$$

In particular

$$P_1(x) = 1$$
$$P_2(x) = x^2 + x - 1$$
$$P_3(x) = x^4 + x^3 + 2x^2 - x + 1$$
$$P_4(x) = x^6 + x^5 + 2x^4 + 3x^3 - 2x^2 + x - 1$$
$$P_5(x) = x^8 + x^7 + 2x^6 + 3x^5 + 5x^4 - 3x^3 + 2x^2 - x + 1.$$

Thus the coefficients of $P\ (x)$ are the first n Fibonacci numbers followed by the reversed sequence with alternating signs.

We shall begin by showing that the roots of $P_n(x)$ lie on two concentric circles in the complex plane. More precisely, we have

Theorem A

The roots of $P_n(x)$ are given explicitly by

$$\alpha\zeta_n^\nu,\ \beta\zeta_n^\nu \quad (\nu = 1, 2, \ldots, n - 1),$$

where

$$\alpha = (1 + \sqrt{5})/2,\ \beta = (1 - \sqrt{5})/2$$

and ζ_n is the nth root of unity $e^{2\pi i/n}$.

Proof: If we multiply $P_n(x)$ by $x^2 - x - 1$, we find that, after collecting the coefficients of $1, x, x^2, \ldots, x^{2n}$, all these coefficients vanish except three, because

$$F_k = F_{k-1} + F_{k-2}.$$

The remaining trinomial is

$$x^{2n} - (F_n + 2F_{n-1})x^n + (-1)^n.$$

Since

$$F_n + 2F_{n-1} = F_{n-1} + F_{n+1} = L_n = \alpha^n + \beta^n,$$

we see at once that

$$(x^2 - x - 1)P_n(x) = x^{2n} - L_n x^n + (-1)^n = x^{2n} - (\alpha^n + \beta^n)x^n + (\alpha^n\beta^n).$$

It is obvious that the quadratic in y obtained by putting $x^n = y$ has for its roots α^n and β^n.

Hence $(x^2 - x - 1)P_n(x)$ has for its roots α, β times all the nth roots of unity. Omitting the extraneous roots α and β, we are left with the $2n - 2$ roots of $P_n(x)$ as specified by the theorem.

As for the sum $S_k(n)$ of the kth powers of the roots of $P_n(x)$, we have

Theorem B

$$S_k(n) = \begin{cases} (n-1)L_k & \text{if } n \text{ divides } k, \\ -L_k & \text{otherwise.} \end{cases}$$

Proof: Using Theorem A, we have

$$S_k(n) = (\alpha^k + \beta^k)\sum_{\nu=1}^{n-1}\zeta_n^{k\nu} = L_k\left(-1 + \sum_{\nu=0}^{n-1}\zeta_n^{k\nu}\right).$$

But if n divides k, then

$$\sum_{\nu=0}^{n-1}\zeta_n^{k\nu} = \sum_{\nu=0}^{n-1} 1 = n,$$

while if n does not divide k,

$$\sum_{\nu=0}^{n-1}\zeta_n^{k\nu} = (1 - (\zeta_n^k)^n)/(1 - \zeta_n^k) = 0.$$

We can make two statements about the factors of the discriminant D of $P_n(x)$, which is the product of all the (nonzero) differences of its roots, namely:

Theorem C

The discriminant D of $P_n(x)$ is divisible by $5^{n-1}n^{2n-4}$.

Proof: Among the differences there are three special types:

$$\alpha(\zeta_n^i - \zeta_n^j);\ \beta(\zeta_n^i - \zeta_n^j);\ \pm(\alpha - \beta)\zeta_n^i \quad (i \neq j = 1, 2, \ldots, n-1).$$

The product of the last type is equal in absolute value to

$$(\alpha - \beta)^{2n-2} = 5^{n-1}.$$

If we allow i and j to be zero, the first two types contribute in absolute value the factor

$$\left[\prod_{i \neq j} |\zeta^i - \zeta^j|\right]^2,$$

which is the square of the discriminant of $x^n - 1$, which is well known to be n^n. If we now remove the product of those differences in which i or j equals zero, we remove

$$\prod_{j=1}^{n-1}(1 - \zeta_n^j)^2 = n^2$$

from the inner product. Hence the theorem.

We now present the following small table of the discriminant of P_n:

n	D
2	5
3	$2^2 \cdot 3^2 \cdot 5^2$
4	$2^8 \cdot 3^2 \cdot 5^3$
5	5^{16}
6	$2^{20} \cdot 3^8 \cdot 5^5$
7	$5^6 \cdot 7^{10} \cdot 13^{10}$

We note that Theorems A and B, as well as their proofs, remain valid if we replace F_n by U_n and L_n by V_n, where

$$U_0 = 0,\ U_1 = 1,\ U_n = Au_{n-1} + U_{n-2}$$
$$V_0 = 1,\ V_1 = A,\ V_n = AV_{n-1} + V_{n-2}$$

and α, β by $(A \pm \sqrt{A^2 + 4})/2$.

◆◇◆◇◆