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INTERSECTIONS OF SECOND-ORDER LINEAR RECURSIVE SEQUENCES

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1. INTRODUCTION

We consider here intersections of positive integer sequences

$$\{w_n(w_0, w_1; p, -q)\}$$

which satisfy the second-order linear recurrence relation

$$w_n = pw_{n-1} + qw_{n-2},$$

where p, q are positive integers, $p \geqslant q$, and which have initial terms w_0, w_1. Many properties of $\{w_n\}$ have been studied by Horadam [2; 3; 4] (and elsewhere), to whom some of the notation is due. We look at conditions for fewer than two intersections, exactly two intersections, and more than two intersections. This is a generalization of work of Stein [5] who applied it to his study of varieties and quasigroups [6] in which he constructed groupoids which satisfied the identity $a((a \cdot ba)a) = b$ but not $(a(ab \cdot a))a = b$.

2. FEWER THAN TWO INTERSECTIONS

We shall first establish some lemmas which will be used to show that two of these generalized Fibonacci sequences with the same p and q generally do not meet.

Suppose the integers a_0, a_1, a_2, a_3, b_0, and b_1 are such that

$$a_2 > b_0 > a_0 \quad \text{and} \quad a_3 > b_1 > a_1.$$

These conditions are not as restrictive as they might appear, although they may require the sequences being compared to be realigned by redefining the initial terms. We consider the sets

$$\{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad \{w_n(b_0, b_1; p, -q)\},$$

and we seek an upper bound L for the number of a_1's $(b_1 > a_1 \geqslant b_0)$ such that

$$\{w_n(a_0, a_1; p, -q)\} \cap \{w_n(b_0, b_1; p, -q)\} \neq \emptyset.$$

We shall show that if $A(b) = b - L$ $(b = b_1 - b_0)$ is the number of a_1's such that if this intersection is nonempty, then $\lim_{b\to\infty} A(b)/b = 1$; that is, these generalized sequences do not meet, because if $\lim_{n\to\infty} A(n)/n = 1$, then we can say that for the predicate P about positive integers n $\{n : P(n)$ is true$\}$ has density 1, which means that P holds "for almost all n."

We first examine where $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ might meet. Since $a_0 < b_0$ and $a_1 < b_1$, then $a_n < b_n$ for all n by induction. Thus, if $a_k \in \{w_n(b_0, b_1; p, -q)\}$ and $a_k = b_i$, then i must be less than k.

Now

$$a_2 > b_0, \text{ and } a_3 > b_1,$$

so that

$$a_4 = pa_3 + qa_2 > pb_1 + qb_0 = b_2, \text{ and so on;}$$

that is,

$$a_k > b_{k-2} \text{ for } k \geqslant 3.$$

Thus, if

$$a_k \in \{w_n(b_0, b_1; p, -q)\},$$

then

$$b_{k-2} < a_k < b_k; \text{ that is, } a_k = b_{k-1}.$$

We next examine the a_1 for which $a_k = b_{k-1}$. Since

$$a_k = a_1 u_{k-1} + qa_0 u_{k-2} \qquad \text{(from (3.14) of [2])}$$

where $\{u_n\} = \{w_n(1, p; p, -q)\}$ is related to Lucas' sequence, then

$$a_k = b_{k-1}$$

is equivalent to

$$b_{k-1} = a_1 u_{k-1} + qa_0 u_{k-2} \quad \text{ or } \quad a_1 = (b_{k-1} - qa_0 u_{k-2})/u_{k-1}.$$

We now define

$$x_k = (b_{k-1} - qa_0 u_{k-2})/u_{k-1},$$

and we shall show that $x_1, x_2, x_3, \ldots$ has a limit X, that it approaches this limit in an oscillating fashion, and that $x_{k+1} - x_k$ approaches zero quickly.

Lemma 1

$$x_{k+1} - x_k = (-q)^{k-1}(b_1 - b_0 - qa_0)/u_k u_{k-1}.$$

Proof: $$x_{k+1} - x_k = \frac{b_k - qa_0 u_{k-1}}{u_k} - \frac{b_{k-1} - qa_0 u_{k-2}}{u_{k-1}}$$

$$= \frac{(b_k u_{k-1} - b_{k-1} u_k) + qa_0(u_k u_{k-2} - u_{k-1}^2)}{u_k u_{k-1}}$$

Now

$$(-q)^{k-1} = u_{k-1}^2 - u_k u_{k-2}, \quad \text{(from (27) of [3])}$$

$$b_k u_{k-1} = b_1 u_{k-1}^2 + q b_0 u_{k-1} u_{k-2}, \quad \text{(from (3.14) of [2])}$$

$$b_{k-1} u_k = b_1 u_k u_{k-2} + q b_0 u_k u_{k-3},$$

so that

$$b_k u_{k-1} - b_{k-1} u_k = b_1 (u_{k-1}^2 - u_k u_{k-2}) + q b_0 (u_{k-1} u_{k-2} - u_k u_{k-3})$$
$$= (-q)^{k-1} b_1 - (-q)^{k-1} b_0$$

since

$$(-q)^{k-2} = u_{k-1} u_{k-2} - u_k u_{k-3} \quad \text{(from 4.21) of [2]).}$$

This gives the required result.

Lemma 2

$|x_{k+1} - x_k| < |b_1 - b_0 - qa_0|/\alpha^{2k-4}$, where α, β, $|\alpha| > |\beta|$, are the roots, assumed distinct, of

$$x^2 - px - q = 0.$$

Proof: $u_k = pu_{k-1} + qu_{k-2} \geqslant pu_{k-1}$

$$\geqslant qu_{k-1} \quad (p \geqslant q)$$
$$\geqslant q^2 u_{k-2} \geqslant \cdots \geqslant q^k u_0 \geqslant q^{k-1}$$

and

$$u_k u_{k-1} > q^{2k-3}.$$

Thus

$$|x_{k+1} - x_k| < |(b_1 - b_0 - qa_0)/q^{k-2}|,$$

which implies that the x_k's converge to a limit X in an oscillating fashion. Now

$$|q|^{k-2} = |\alpha|^{k-2}|\beta|^{k-2} < \alpha^{2k-4},$$

and

$$|x_{k+1} - x_k| < |b_1 - b_0 - qa_0|/\alpha^{2k-4}.$$

Theorem 1

If a_0 is a positive integer and $\{w_n\}$ is a generalized Fibonacci sequence, then for almost all a_1, $\{w_n(a_0, a_1; p, -q)\} \cap \{w_n\}$ consists of at most the element a_0.

Proof: It follows from Lemma 2 that at most one x_k is an integer for those k which satisfy the inequality

$$(b_1 - b_0 - qa_0)/\alpha^{2k-4} < 1,$$

or, equivalently, the inequality

$$k > 2 + \underline{\log}(b_1 - b_0 - qa_0)^{1/2}$$

in which $\underline{\log}$ stands for logarithm to the base $|\alpha|$. Thus the total number of k's for which x_k is an integer (since a_1 must be an integer) is at most

$$L = 2 + \underline{\log}(b_1 - b_0 - qa_0)^{1/2}.$$

If we choose b_0 such that $b_0 = c_m$ and $b_1 = c_{m+1}$, $c_m \varepsilon \{w_n(c_0, c_1; p, -q)\}$, where $c_{m+1}/c_m < [1+\alpha]$, then L is small in comparison with $b - b_0$. There is such an integer m:

$$c_{m+1}/c_m < [1 + \alpha] \qquad \text{for all } k \geqslant m$$

since

$$\lim_{k \to \infty} c_{k+1}/c_k = \alpha. \qquad ((1.22) \text{ of } [4])$$

We could take $b_0 = c_{m+1}$ or c_{m+2} and still conclude that the total number of a_1's ($b_0 \leqslant a_1 < b_1$) for which $\{w_n(a_0, a_1; p, -q)\}$ meets $\{w_n(b_0, b_1; p, -q)\}$ is small in comparison with $b = b_1 - b_0$.

Thus

$$A(b) = b - L,$$

and since

$$\lim_{b \to \infty}(\underline{\log}\, b)/b = 0,$$

we have

$$\lim_{b \to \infty} A(b)/b = 1 - \lim_{b \to \infty}(2 + \underline{\log}(b - qa_0)^{1/2})/b$$

$$= 1, \text{ as required.}$$

Thus, for allmost all a_1, $\{w_n\} \cap \{w_n(a_0, a_1; p, -q)\}$ contains a_0 only or is empty.

3. EXACTLY TWO INTERSECTIONS

Lemma 3

If $a_i = b_j$ and $a_{i-1} \neq b_{j-1}$, then for $r \geqslant 1$

$$b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{i+r} \notin \{w_n(b_0, b_1; p, -q)\}.$$

Proof: If $a_{i-1} > b_{j-1}$, then $a_{i+1} > b_{j+1}$, and

$$a_{i+1} = pa_i + qa_{i-1} < pb_{j+1} + qb_j = b_{j+2},$$

since

$$a_{i-1} < a_i = b_j < b_{j+1}$$

Thus

$$a_i < b_{j+1} < a_{i+1} \quad \text{and} \quad a_{i+1} < b_{j+2} < a_{i+2},$$

and, by induction,

$$a_{i+r-1} < b_{j+r} < a_{i+r} \qquad (r \geqslant 1).$$

Hence, $b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\}$, $r \geqslant 1$, from which the lemma follows.

Theorem 2

If $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ meet exactly twice, then at least one of these statements holds:

$$a_0 \in \{w_n(b_0, b_1; p, -q)\}, \; b_0 \in \{w_n(a_0, a_1; p, -q)\}.$$

As an illustration of Theorem 2, consider the sequences

$$1, 4, 5, 9, 14, \ldots, \quad \text{and} \quad 1, 1, 2, 3, 5, 8, 13, \ldots;$$

the second of these is the sequence of ordinary Fibonacci numbers

$$\{w_n(1, 1; 1, -1)\}.$$

Proof of Theorem 2: If $a_i = b_j$, $i, j > 0$, and the sequences meet exactly twice, then $a_{i-1} \neq b_{j-1}$; otherwise the sequences would be identical from those terms on, as can be seen from Theorem 3. (We need $i, j > 0$, since we have not specified a_n, b_n for $n < 0$.) Thus, from Lemma 3,

$$b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{i+r} \notin \{w_n(b_0, b_1; p, -q)\}, \; r \geqslant 1.$$

So $a_n = b_m$, $0 < m < j$, $0 < n < i$, and, again, $a_{n-1} \neq b_{m-1}$; otherwise the sequences would be identical from those terms on. But from Lemma 3 this implies that

$$b_{m+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{n+r} \notin \{w_n(b_0, b_1; p, -q)\}, \; r \geqslant 1,$$

which contradicts the assumption that $a_i = b_j$. So the only other possibilities are that $a_0 = b_m$ for some m or $a_n = b_0$ for some n, as required. This establishes the theorem.

4. MORE THAN TWO INTERSECTIONS

Theorem 3

If $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ have two consecutive terms equal, then they are identical from those terms on.

Proof: If $a_i = b_j$ and $a_{i-1} = b_{j-1}$, then

$$a_{i+1} = pa_i + qa_{i-1} = pb_j + qb_{j-1} = b_{j+1}$$

and the result follows by induction.

5. REMARKS

A. It is of interest to note that the number of terms of $\{w_n(a_0, a_1; p, -q)\}$ not exceeding b_0 is asymptotic to

$$\underline{\log}(b_0(\alpha - \beta)/(a_1\alpha + a_0\alpha\beta)). \qquad \text{(Horadam [4])}$$

B. As an illustration of Theorem 1, if we consider the case where $p = q = 1$, and if we take $a_0 = 1$, $b_0 = 100$, $b_1 = 191$, then $b_2 = 291$, $b_3 = 392$, $b_4 = 683$. When:

$$a_1 = 100,\ a_1 = b_0; \quad a_1 = 190,\ a_2 = b_1; \quad a_1 = 145,\ a_3 = b_2;$$

$$a_1 = 130,\ a_4 = b_3; \quad a_1 = 136,\ a_5 = b_4.$$

Thereafter, there are no more integer values of a_1 that yield $a_k = b_{k-1}$. Thus 100, 130, 136, 145, and 190 are the only values of a_1 $(100 \leqslant a_1 < 191)$ for which

$$\{w_n(1, a_1; 1, -1)\} \cap \{w_n(100, 191; 1, -1)\} \neq \emptyset.$$

Also, $\left[\left(\frac{1}{2}(4 + \underline{\log}\ 90)\right)\right] = 6$, so the bound L is valid.

C. It is not apparent how Theorem 1 can be elegantly generalized to arbitrary order sequences. If $\{w_n^{(r)}\}$ satisfies the recurrence relation

$$w_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n-j}^{(r)} \qquad n \geqslant r$$

with suitable initial values, where the P_{rj} are arbitrary integers, and if $\{u_n^{(r)}\}$ satisfies the same recurrence relation, but has initial values given by

$$u_0^{(r)} = u_1^{(r)} = \cdots = u_{r-2}^{(r)} = 0,\ u_{r-1}^{(r)} = 1,$$

then it can be proved that

$$w_n^{(r)} = \sum_{j=0}^{r-1} \left(\sum_{k=0}^{j} (-1)^{j-k} P_{rj}\, w_k^{(r)} \right) u_{n-j+1}^{(r)},$$

where $P_{r0} = 1$. When $r = 2$, this becomes

$$\begin{aligned} w_n^{(2)} &= w_1^{(2)} u_n^{(2)} + w_0^{(2)}\ u_{n+1}^{(2)} - P_{21} u_n^{(2)} \\ &= w_1^{(2)} u_n^{(2)} - P_{22} w_0^{(2)} u_{n-1}^{(2)} \end{aligned}$$

which is Eq. (3.14) of [2] for the sequences

$$\{w_n^{(2)}\} = \{w_n(w_0^{(2)}, w_1^{(2)}; P_{21}, P_{22})\}$$

and

$$\{u_{n+1}^{(2)}\} = \{w_n(1, P_{21}; P_{21}, P_{22})\}.$$

Thus, one of the key equations in Theorem 1 generalizes to

$$w_{r-1}^{(r)} = \left(w_n^{(r)} - \sum_{j=0}^{r-2} (-1)^{j-r-1} P_{r,\, r-j-1} w_j^{(r)} u_{n-r+2}^{(r)} + \sum_{k=0}^{j} (-1)^{j-k} P_{r,j-k}\, w_k^{(r)} u_{n-j+1}^{(r)}\right) \Big/ u_{n-r+2}^{(r)},$$

which is rather cumbersome.

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REFERENCES

1. A. F. Horadam. "A Generalized Fibonacci Sequence." *American Mathematical Monthly* 68 (1961):455-459.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *Fibonacci Quarterly* 3 (1965):161-176.
3. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* 32 (1965):437-446.
4. A. F. Horadam. "Generalizations of Two Theorems of K. Subba Rao." *Bull. Calcutta Math. Soc.* 58 (1966):23-29.
5. S. K. Stein. "The Intersection of Fibonacci Sequences." *Mich. Math. J.* 9 (1962):399-402.
6. S. K. Stein. "Finite Models of Identities." *Proc. Amer. Math. Soc.* 14 (1963):216-222.

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