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GENERALIZED PROFILE NUMBERS

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(Submitted December 1981)

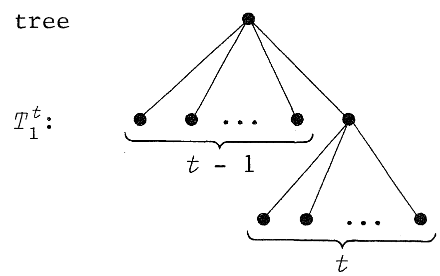
INTRODUCTION

A family of binary trees $\{T_i\}$ is studied in [2]. The numbers $p(n, k)$ of internal nodes on level k in T_n (the root is considered to be on level 0) are called profile numbers, and they "enjoy a number of features that are strikingly similar to properties of binomial coefficients" (from [2]). We extend the results in [2] to t-ary trees.

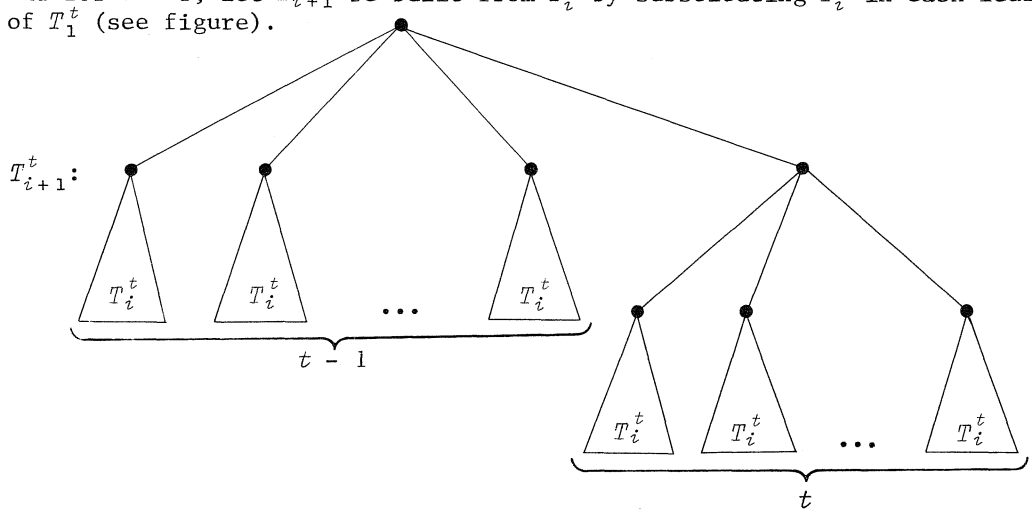
DISCUSSION

We discuss t-ary trees (see Knuth [1]). A t-ary tree either consists of a single root, or a root that has t ordered sons, each being a root of another t-ary tree.

Let T_1^t be the tree



and for $i \geqslant 1$, let T_{i+1}^t be built from T_i^t by substituting T_i^t in each leaf of T_1^t (see figure).



Let $p_t(n, k)$ denote the number of internal nodes at level k in the tree T_n^t.

The numbers $p_t(n, k)$ satisfy the recurrence relation

$$p_t(n+1, k+1) = (t-1)p_t(n, k) + t\,p_t(n, k-1) \tag{1}$$

together with the boundary conditions

$$\begin{aligned} p_t(n, 0) &= 1 \\ p_t(1, 1) &= 1 \\ p_t(n, 1) &= t \quad \text{for } n > 1 \\ p_t(1, k) &= 0 \quad \text{for } k > 1. \end{aligned} \tag{2}$$

The corresponding trees and sequences for the case of binary trees ($t = 2$) is studied in [2]. Thus, T_n and $p(n, k)$ in [2] are denoted here by T_n^2 and $p_2(n, k)$, respectively.

We first show that

$$p_t(n, k) = t^{k-n} \sum_{0 \leqslant i < 2n-k} (t-1)^i \binom{n}{i}, \tag{3}$$

where $n \geqslant 1$, $k \geqslant 0$, and the $\binom{n}{i}$'s are the binomial coefficients.

Note that when $k < n$ we have $p_t(n, k) = t^k$.

The expression in (3) is easily shown to satisfy the boundary conditions (2). To continue, we induct on n (and arbitrary k); using (1) and the inductive hypothesis, we get

$$\begin{aligned} p_t(n+1, k+1) &= (t-1)p_t(n, k) + t\,p_t(n, k-1) \\ &= t^{k-n} \sum_{0 \leqslant i < 2n-k} (t-1)^{i+1}\binom{n}{i} + t^{k-n} \sum_{0 \leqslant i < 2n-k+1} (t-1)^i \binom{n}{i} \\ &= t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n}{i-1} + t^{k-n} + t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n}{i} \\ &= t^{k-n} + t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n+1}{i} = t^{k-n} \sum_{0 \leqslant i < 2n-k+1} (t-1)^i \binom{n+1}{i} \end{aligned}$$

and this establishes (3).

Using (3), we get

$$p_t(n, k+1) = t\,p_t(n, k) - t^{k-n+1}(t-1)^{2n-k-1}\binom{n}{k-n+1} \tag{4}$$

and

$$p_t(n+1,\ k) = p_t(n,\ k) + t^{k-n-1}(t-1)^{2n-k}\left[\binom{n}{k-n} + (t-1)\binom{n+1}{k-n}\right] \quad (5)$$

where $n \geqslant 1$ and $k \geqslant 0$.

Let x_n^t be the number of internal nodes in T_n^t, namely

$$x_n^t = \sum_{0 \leqslant k < 2n} p_t(n,\ k). \quad (6)$$

Using (3), changing the order of summation, and applying the binomial theorem results in

$$x_n^t = \frac{(2t-1)^n - 1}{t-1}. \quad (7)$$

Note that, by their definition, the numbers x_n^t satisfy the recurrence relation

$$x_1^t = 2$$

$$x_{i+1}^t = (2t-1)x_i^t + 2 \quad \text{for} \quad i > 0, \quad (8)$$

which also implies (7).

Let ℓ_n^t denote the internal path length (see [1]) of T_n^t, namely

$$\ell_n^t = \sum_{0 \leqslant k < 2n} k\,p_t(n,\ k). \quad (9)$$

The numbers ℓ_n^t also satisfy the recurrence relation

$$\ell_1^t = 1$$

$$\ell_{i+1}^t = (2t-1)\ell_i + (3t-1)x_i + 1 \quad \text{for} \quad i > 0. \quad (10)$$

Using (9) and (3), or solving (10) with the use of (7), one gets

$$\ell_n^t = \frac{3t-1}{t-1}\,n(2t-1)^{n-1} - \frac{t}{(t-1)^2}((2t-1)^n - 1). \quad (11)$$

The average level e_n^t of a node in T_n^t is thus given by ℓ_n^t/x_n^t, and satisfies

$$e_n^t \approx \frac{3t-1}{2t-1}\,n + 0(1). \quad (12)$$

The results in (1), (2), (3), (4), (5), (7), and (11) are extensions of (1), (3), Theorems 1, 2a, 2b, 3, and 4 of [2], respectively.

If we denote

$$F_t(x, y) = \sum_{n \geq 1,\, k \geq 0} p_t(n, k) x^n y^k,$$

then, using (1) and (2), we get

$$F_t(x, y) = \frac{x(1 + y)}{(1 - x)(1 - txy + xy - txy^2)}. \tag{13}$$

Equations (1) and (7), for the case $t = 2$, were noted in [2] to be similar to the recurrence relation

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

and the summation formula

$$\sum_{0 \leq k < n} \binom{n}{k} = 2^n - 1.$$

The binomial coefficients also satisfy

$$\sum_k (-1)^k \binom{n}{k} = 0.$$

Using (3), one can show that the same identity holds for any t and n; namely,

$$\sum_{0 \leq k < 2n} (-1)^k p_t(n, k) = 0. \tag{14}$$

REFERENCES

1. D. E. Knuth. *The Art of Computer Programming*. Vol. I: *Fundamental Algorithms*. New York: Addison-Wesley, 1968.
2. A. L. Rosenberg. "Profile Numbers." *Fibonacci Quarterly* 17 (1979): 259-264.

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