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A FAMILY OF POLYNOMIALS AND POWERS OF THE SECANT

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In this paper, we discuss a family of polynomials $A_n(z)$, defined by the conditions

$$A_0(z) = 1 \quad \text{and} \quad A_k(z) = z(z+1)A_{k-1}(z+2) - z^2A_{k-1}(z).$$

Using these polynomials, we may express complex powers of the secant and cosine functions as infinite series. These polynomials provide ways to obtain numerous relations among Euler numbers and Bell numbers. They appear to be unrelated to other functions which arise in this context.

Suppose that we consider the family of polynomials $A_n(z)$, $n = 0, 1, 2, \ldots$, defined as follows: $A_0(z) = 1$ and if $A_0(z), \ldots, A_{k-1}(z)$ have already been defined, then $A_k(z)$ is given by the recursion formula:

$$A_k(z) = z(z+1)A_{k-1}(z+2) - z^2A_{k-1}(z). \tag{1}$$

It follows immediately that if $A_\ell(z)$ is a polynomial of degree ℓ for $0 \leqslant \ell \leqslant k-1$, then $A_k(z)$ has leading coefficient $(2k-1)a_{k-1}$, where a_{k-1} is the leading coefficient of $A_{k-1}(z)$, and where $(2k-1)a_{k-1}$ is the coefficient of z_k. Thus, we generate a series of leading terms:

$$1,\ z,\ 3z^2,\ 15z^3,\ 105z^4,\ \ldots,\ [(2k-1)!/2^{k-1}(k-1)!]z^k,\ \ldots, \tag{2}$$

so that $A_k(z)$ is a polynomial of degree precisely k. It also follows immediately from (1) that $A_k(0) = 0$ for all $k > 1$. We note further that if $A_k^*(z) = A_k(-z)$, then $A_0^*(z) = A_0(-z) = 1$, and from (1), $A_k^*(z) = A_k(-z) = -z(-z+1)A_{k-1}^*(z-2) - z^2A_{k-1}^*(z)$ so that we have a corresponding family of polynomials $A_n^*(z)$, $n = 0, 1, 2, \ldots$, given by the recursion formula:

$$A_k^*(z) = z(z-1)A_{k-1}^*(z-2) - z^2A_{k-1}^*(z). \tag{3}$$

It follows immediately that for the sequence $A_n^*(z)$ we have a corresponding sequence of leading terms:

$$1,\ -z,\ 3z^2,\ -15z^3,\ 105z^4,\ \ldots\ [(-1)^k(2k-1)!/2^{k-1}(k-1)!]z^k,\ \ldots \tag{4}$$

It is our purpose in this note to prove that

$$\sec^z x = \sum_{n=0}^{\infty} [A_n(z)/(2n)!]x^{2n} \tag{5}$$

and

$$\cos^z x = \sum_{n=0}^{\infty} [A_n^*(z)/(2n)!]x^{2n}, \tag{6}$$

as well as derive some consequences of these facts.

In particular, if $z = 1$, then we obtain the corresponding formulas,

$$\sec x = \sum_{n=0}^{\infty} [A_n(1)/(2n)!]x^{2n} \tag{7}$$

and

$$\cos x = \sum_{n=0}^{\infty} [A_n^*(1)/(2n)!]x^{2n}, \tag{8}$$

so that we obtain the results: $A_n(1) = E_{2n}$, the usual Euler number; and $A_n^*(1) = A_n(-1) = (-1)^n$, so that we are able to evaluate these polynomials at these values by use of the definitions.

Given that formulas (5) and (6) hold, we obtain from

$$\sec^{z_1} x \cdot \sec^{z_2} x = \sec^{z_1+z_2} x$$

the relation

$$\begin{aligned}\left(\sum_{m=0}^{\infty} [A_m(z_1)/(2m)!]x^{2m}\right)\left(\sum_{\ell=0}^{\infty} [A_\ell(z_2)/(2\ell)!]x^{2\ell}\right)\\ = \sum_{k=0}^{\infty}\left(\sum_{m+\ell=k} A_m(z_1)A_\ell(z_2)/(2m)!(2\ell)!\right)x^{2k};\end{aligned} \tag{9}$$

whence,

$$\sum_{m+\ell=k} A_m(z_1)A_\ell(z_2)/(2m)!(2\ell)! = A_k(z_1+z_2)/(2k)!, \tag{10}$$

so that we obtain finally the addition formula:

$$A_k(z_1+z_2) = \sum_{j=0}^{k}\binom{2k}{2j}A_j(z_1)A_{k-j}(z_2). \tag{11}$$

From (11) we have the consequence

$$A_k(z-z) = \sum_{j=0}^{k}\binom{2k}{2j}A_j(z)A_{k-j}^*(z) = 0,\ k > 0; \tag{12}$$

whence, since $\sec^z x \cdot \cos^z x = 1$, it follow that for $k \geqslant 1$,

$$\sum_{j=0}^{k} \binom{2k}{2j} A_j(z) A^*_{k-j}(z) = 0. \tag{13}$$

In particular, if $z = 1$, then we obtain the formula for $k \geqslant 1$, using $A^*_{k-j}(-1) = (-1)^{k-j}$,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} E_{2j} = 0. \tag{14}$$

To generate sample polynomials, we use the original relations (1) and take consecutive values of k,

$$k = 1,\ A_1(z) = z(z+1) - z^2 = z$$

$$k = 2,\ A_2(z) = z(z+1)(z+2) - z^3 = 3z^2 + 2z$$

$$\begin{aligned} k = 3,\ A_3(z) &= z(z+1)[3(z+2)^2 + 2(z+2) - z^2(3z^2 + 2z) \\ &= 15z^3 + 30z^2 + 16z, \text{ etc.,} \end{aligned}$$

with $A_1(1) = E_2 = 1$, $A_2(1) = E_4 = 5$, $A_3(1) = E_6 = 61$.

From Equation (1) we find, taking $z = 1$, that

$$E_{2k} = 2A_{k-1}(3) - E_{2k-2},\ E_0 = 1 \tag{15}$$

and

$$A_{k-1}(3) = 1/2[E_{2k} + E_{2k-2}],\ k \geqslant 1, \tag{16}$$

so that we have an immediate expansion for $\sec^3 z$ in terms of the Euler numbers:

$$\begin{aligned} \sec^3 x &= \sum_{n=1}^{\infty} ([E_{2n} + E_{2n-2}]/2(2n-2)!)x^{2n-2} \\ &= \sum_{n=0}^{\infty} [E_{2n+2} + E_{2n}]/2(2n)!x^{2n}. \end{aligned} \tag{17}$$

By repeated use of (1) in this fashion, we may generate expressions for $\sec^5 z$, $\sec^7 z$, ..., $\sec^{2n+1} z$, which are expressed in terms of the standard Euler numbers only.

To prove the formulas (5) and (6), we proceed as follows:

$$(\sec^m x)' = m \sec^m x \tan x,$$

$$(\sec^m x)'' = (m^2 + m)\sec^{m+2} x - m^2 \sec^m x,$$

and thus, if we write (formally)

$$\sec^m x = \sum_{n=0}^{\infty} [A_n(m)/(2n)!]x^{2n},$$

then

$$(m^2 + m)\sec^{m+2}x - m^2\sec^m x = \sum_{n=0}^{\infty} [((m^2 + m)A_n(m + 2) - m^2A_n(m))/(2n)!]x^{2n}$$

$$= \sum_{n=0}^{\infty} [A_{n+1}(m)/(2n)!]x^{2n}, \tag{18}$$

so that upon equating coefficients, we find:

$$A_{n+1}(m) = m(m + 1)A_n(m + 2) - m^2A_n(m). \tag{19}$$

From (19), it is immediate that $A_k(m)$ is a polynomial in the variable m, where we consider m a real number $m > 1$, and such that $\sec^m x$ has the appropriate expression.

If we fix x so that $\sec^z x > 1$, then $f(z) = \sec x$ yields

$$f'(z) = f(z) \cdot \log(\sec x),$$

and thus $f(z)$ is an analytic function of z which agrees with the series given in (5) for the real variable $m > 1$. Since $g(z)$ given by the series in z is also analytic and since $f(m) = g(m)$ for the real variable $m > 1$, it follows that $f(z) = g(z)$, or what amounts to the same thing, equation (5) holds for all z. Equation (6) is now a consequence of equation (5) if we replace z by $-z$.

Making use of what we have derived above, we may also analyze other functions in this way, as the examples below indicate.

Suppose we write

$$\begin{aligned} \tan x &= \sum_{n=0}^{\infty} (T_n/n!)x^n, \text{ where } T_{2n} = 0 \text{ and} \\ T_{2n-1} &= \frac{(-1)^{n-1}2^{2n}(2^{n-1} - 1)B_{2n}}{(2n)!}. \end{aligned} \tag{20}$$

Then from $\frac{d}{dx}(\sec x) = z \sec^z x \tan^z x$ we obtain the relation

$$z\left[\sum_{n=0}^{\infty} \left\{\frac{A_n(z)/z}{(2n - 1)!}\right\} x^{2n-1}\right] = z\left[\sum_{m=0}^{\infty} \frac{A_m(z)}{(2m)!} x^{2m}\right]\left[\sum_{\ell=0}^{\infty} T_\ell/\ell! x^{2\ell}\right], \tag{21}$$

whence it follows that:

$$\frac{A_n(z)/z}{(2n-1)!} = \sum_{2m+\ell=2n-1} \frac{A_m(z)T_\ell}{(2m)!\ell!} \tag{22}$$

or

$$A_n(z)/z = \sum_{2m+\ell=2n-1} \binom{2n-1}{\ell} A_m(z)T_\ell. \tag{23}$$

In particular, we conclude that if ℓ is even, then $T_\ell = 0$, which is of course known, and the corresponding expression is

$$A_n(z) = \sum_{m=0}^{n-1} \binom{2n-1}{2m} T_{(2(n-m)-1)} zA_m(z). \tag{24}$$

Hence we may derive a variety of formulas. For example, by taking $z = 1$, (24) yields

$$E_{2n} = \sum_{m=0}^{n-1} \binom{2n-1}{2m} T_{2(n-m)-1} E_{2m}; \tag{25}$$

or, since the coefficients T_ℓ are vastly more complicated:

$$T_{2n-1} = E_{2m} - \sum_{m=1}^{n-1} \binom{2n-1}{2m} T_{2(n-m)-1} E_{2m}, \tag{26}$$

which yields a recursion formula involving the Euler numbers.

Similarly, from $z = -1$, $A_n(-1) = (-1)^n$, we obtain

$$(-1)^n = \sum_{m=0}^{n-1} (-1)^{m+1} \binom{2n-1}{2m} T_{2(n-m)-1}, \tag{27}$$

or, once again, for $m = 0$,

$$T_{2n-1} = \sum_{m=1}^{n-1} (-1)^{m+1} \binom{2n-1}{2m} T_{2(n-m)-1} + (-1)^{n+1}. \tag{28}$$

Using the fact that $1 + \tan^2 x = \sec^2 x$, we obtain the relation

$$\begin{aligned} &\left[\sum_{m=0}^{\infty} \frac{T_{2m+1}}{(2m+1)!} x^{2m+1}\right]\left[\sum_{\ell=0}^{\infty} \frac{T_{2\ell+1}}{(2\ell+1)!} x^{2\ell+1}\right] \\ &= \sum_{k=1}^{\infty}\left(\sum_{\ell+m=k-1} \frac{T_{2m+1}}{(2m+1)!} \frac{T_{2\ell+1}}{(2\ell+1)!}\right) x^{2k}; \end{aligned} \tag{29}$$

so that

$$\sum_{\ell+m=k-1} \frac{T_{2m+1}}{(2m+1)!} \frac{T_{2\ell+1}}{(2\ell+1)!} = \frac{A_k(2)}{(2k)!} \tag{30}$$

and

$$A_k(2) = \sum_{m=0}^{k-1} \binom{2k}{2m+1} T_{2m+1} T_{2(k-m)-1}. \tag{31}$$

If we use the fact that $(\tan x)' = \sec^2 x$, then

$$\sum_{m=0}^{\infty} \frac{T_{2m+1}}{(2m)!} x^{2m} = \sum_{m=0}^{\infty} \frac{A_m(2)}{(2m)!} x^{2m}, \tag{32}$$

so that immediately:

$$A_m(2) = T_{2m+1}. \tag{33}$$

Hence, by using (31), we have the relation:

$$T_{2k+1} = \sum_{m=0}^{k-1} \binom{2k}{2m+1} T_{2m+1} \cdot T_{2(k-m)-1}. \tag{34}$$

Having these relations at hand, we use the fact that

$$\tan x \cdot \cos x = \sin x$$

to obtain

$$\begin{aligned} \sin x &= \left[\sum_{m=0}^{\infty} (A_m(2)/(2m+1)!)x^{2m+1}\right]\left[\sum_{\ell=0}^{\infty} A_\ell(-1)/(2\ell)!x^{2\ell}\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{\ell+m=k} \frac{A_m(2) \cdot A_\ell(-1)}{(2m+1)!(2\ell)!}\right]x^{2k+1} \end{aligned} \tag{35}$$

so that

$$\sum_{\ell+m=k}^{\infty} \frac{A_m(2) \cdot A_\ell(-1)}{(2m+1)!(2\ell)!} = \frac{(-1)^k}{(2k+1)!}. \tag{36}$$

Hence:

$$\sum_{m=0}^{\infty} \binom{2k+1}{2m+1} A_m(2) A_{k-m}(-1) = (-1)^k. \tag{37}$$

Using the fact that $A_\ell(-1) = (-1)^\ell$, it follows that:

$$\sum_{n=0}^{k} (-1)^{2k-m} \binom{2k+1}{2m+1} A_m(2) = 1. \tag{38}$$

From these examples, it should be clear that the polynomials $A_n(z)$, $n = 0, 1, 2, \ldots$ are a family closely related to the trigonometric functions and, hence, they should prove interesting. The sampling of such properties given here seems to indicate that this is indeed the case.

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