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SEQUENCES GENERATED BY SELF-REPLICATING SYSTEMS

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INTRODUCTION

In the late 1940's John von Neumann began to develop a theory of automa. His substantial and unique works covered a broad range of subjects from which one, self-replication, is of particular interest in this study.

A self-replicating system (SRS) is an organization of system elements that is capable of producing exact replicas of itself which, in turn, will produce exact replicas of themselves. The replication process uses materials or components from its environment and continues automatically until the process is terminated. Examples of potential space and terrestrial applications are in the areas of photoelectric cells, oxygen, planetary explorer rovers, ocean bottom mining, and desert irrigation. We will investigate an aspect of SRS's, that which concerns the number of replicas various systems would produce.

OUTLINE FOR A SELF-REPLICATING SYSTEM

For a description of self-replication, the reader sbould refer to [1]. The basic system elements of an SRS are:

- Mining and Materials Processing Plant
- Materials Depot
- Parts Production Plant
- Replication Parts Depot
- Production Parts Depot
- Production Facility
- Universal Constructor
- Product Depot
- Product Retrieval System
- Energy System

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In the Mining and Materials Processing Plant, raw materials are gathered by mining, analyzed, separated, and processed into feedstock such as sheets, bars, ingots, and castings. The processed feedstock is then laid out and stored in the Materials Depot.

The Parts Production Plant selects and transports feedstock from the Materials Depot and produces all parts required for SRS replication and the products. The finished parts are laid out and stored in either the Replication Parts Depot or the Production Parts Depot. The Parts Production Plant includes material transport and distribution, production, control, and sub-assembly operations. All parts and sub-assemblies required for replication of complete SRS's are stored in the Replication Parts Depot in lots destined for specific facility construction. In the Production Parts Depot, parts are stored for use in manufacturing the desired products in the Production Facility.

The Production Facility produces the product. Parts and sub-assemblies are picked up from the Production Parts Depot, transported into the Production Facility, and undergo specific manufacturing and production processes depending on the specific product desired. The finished products are stored in the Product Depot to await pickup by the Product Retrieval System.

The Universal Constructor, in principle, is a system capable of constructing and system. The purpose of the Universal Constructor is to self-replicate a complete SRS a specified number of times in such a way that these replicas, in turn, construct replicas of themselves, and so on. The Universal Constructor has the overall control and command function for its own SRS as well as for the replicas until control and command functions have been replicated and transferred to the replicas. The Product Retrieval System collects the outputs of all units of an SRS field. Finally, the energy source generally considered practical is solar.

SELF-REPLICATING OPTIONS

There are several possible schemes that one must consider in designing a self-replicating system. One is to design each replica to reproduce simultaneously its n-replicas (Figure 1), and we will refer to this

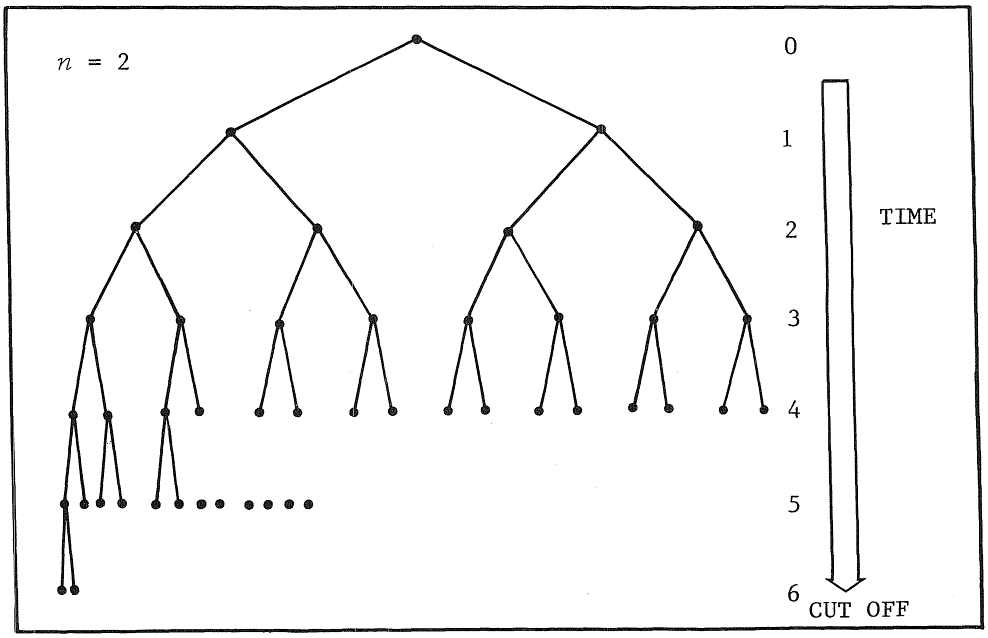

FIGURE 1. Option A, S = 127

case as Option A. Because of large mass flows and programming complexities, this option presently has little support. Another scheme, referred to as Option B, is to design each replica to produce its n-replicas sequentially (Figure 2). When a sufficient number have been obtained, reproduction is stopped and production begins. The main reason for limiting the number of replicas to, say n, is that with each replication a defective replica becomes more likely. An objection to Option B is that earlier branches will have reproduced more generations than later ones, which would result in some lower-quality replicas than necessary.

Therefore, a third scheme (Figure 3) is considered and referred to as Option C: a replica reproduces sequentially no more than n replicas and

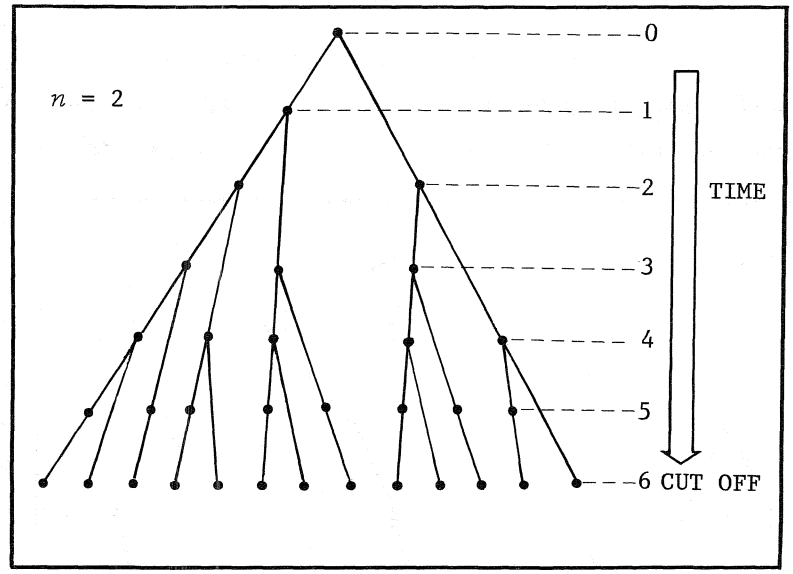

FIGURE 2. Option B, S = 33

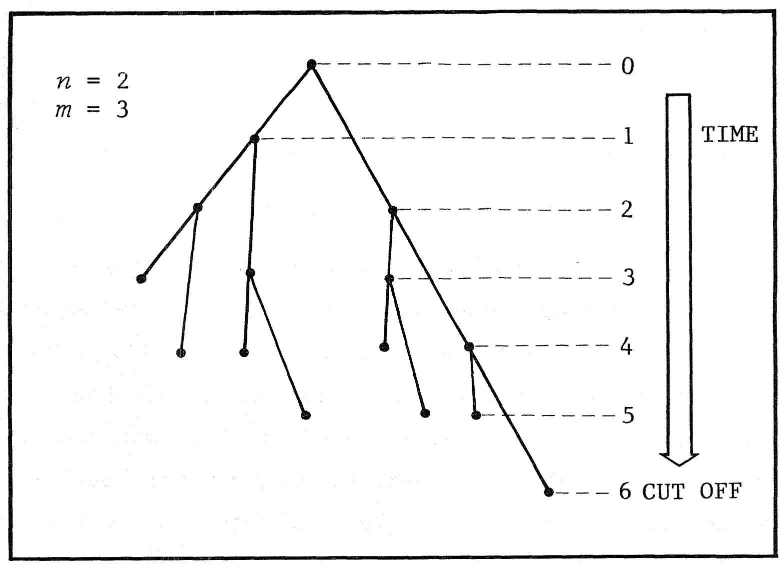

FIGURE 3. Option C, S = 15

in such a way that none will have more than m direct ancestors. We have given a comparison of growth rates between the three options (Figure 4) for the case of two replicas per primary and a limit of three ancestors in case of Option C.

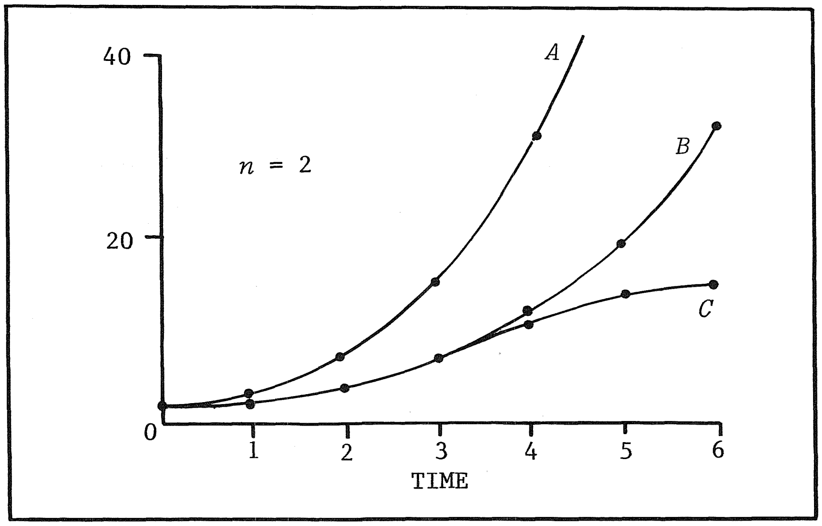

FIGURE 4. Growth Rate Comparison

COMPUTATIONAL ASPECTS OF SELF-REPLICATING SYSTEMS

There are multitudes of novel relationships that one may discover hidden in replicating sequences. We begin by looking at Option A, where replication continues throughout the system until cutoff. The number of replicas s_k generated in the kth time interval is clearly $s_k = n^k$ and accumulates to

$$S_k = \sum_{j=0}^{k} s_k = \frac{n^{k+1} - 1}{n - 1}, \tag{1}$$

so that this option triggers little mathematical curiosity.

In consideration of Option B, we begin with the case of two replications per primary, $n = 2$, and refer to Figure 3. Because each replica produces two offspring, one in each of the two time frames immediately following its own existence, any replica must have come from one of the two previous time frames. This means that the number of replicas s_i produced in the ith time interval equals that produced in the previous two,

$$s_i = s_{i-1} + s_{i-2}. \tag{2}$$

This recursion relation, with $s_0 = s_1 = 1$, gives precisely the Fibonacci numbers. By addition, one computes the total number of replicas S_k in k time intervals to be

$$S_k = s_{k+2} - 1. \tag{3}$$

An explicit formula for s_k is well known, since equation (2) holds:

$$s_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]. \tag{4}$$

Equations (3) and (4) give a formula for the cumulative replicas:

$$S_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+3}\right] - 1. \tag{5}$$

For n replicas per primary

$$s_k = \sum_{i=1}^{n} s_{k-i},$$

s_k and S_k may be calculated by division [2],

$$\frac{1}{1 - \sum_{k=1}^{n} x^k} = \sum_{k=0}^{\infty} s_k x^k \quad \text{and} \quad \frac{1}{(1-x)\left(1 - \sum_{k=1}^{n} x^k\right)} = \sum_{k=0}^{\infty} S_k x^k. \tag{6}$$

Because of linearity, a matrix method can be applied to this problem. To cast Option B with n replicas per primary in the framework of [3], we consider $n + 1$ types of individuals (replicas) denoted by 0, 1, ..., n; the index referring to the number of offsprings this individual has reproduced. One then sets up an $n+1$ by $n+1$ matrix $F = (f_{ij})$, where each individual of type i in the kth time frame gives rise to f_{ij} individuals of type j in the $(k + 1)$th time frame $(1 \leqslant i, j \leqslant n + 1)$ and $k = 0, 1, \ldots$. If the vector $\mathbf{f}(k)$ is the state of the replicas at time k, then $\mathbf{f}(k)F = \mathbf{f}(k+1)$ and by induction $\mathbf{f}(0)F^k = \mathbf{f}(k)$. This means that once we have the matrix F, we can determine the replica state at any future time by matrix multiplication.

For example, let the number of offsprings per replica be 3, $n = 3$. An individual of type i produces a type $i + 1$ and a type 0 if it has reproduced less than 3 and remains a type i if $i = 3$. So if $i < 3$, $f_{i0} = 1$ and $f_{i\,i+1} = 1$, if $i = 3$, $f_{33} = 1$; and $f_{ij} = 0$ otherwise.

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and after four time frames,

$$\mathbf{f}(4) = (1, 0, 0, 0) \quad F^4 = (7, 4, 2, 2);$$

starting with one new replica we have seven with no offspring, four with 1, two with 2, and two with 3 for a total of fifteen.

We now turn to Option C, where the number of replicas is restricted to a fixed number m of generations. In the case where $n = 2$, $m = 3$ (see Figure 2), one observes that the diagram is the same as Option B until the limited number of generations begins to curtail replication; equality ceases after $k = 3$. One observes also that adding one more generation would add two replicas for each with maximal m ancestors; this would add a total of 2^m replicas and, in general, n^m replicas. We find the sum for m generations by adding the terms:

$$S = \sum_{j=0}^{m} n^j = \frac{n^{m+1} - 1}{n - 1}. \tag{7}$$

Again, we are able to use the matrix method [3] to find the state of the replicas at any time. Two indices, a and b, are used to denote the type of replica; the first for the number of offsprings, the second for its ancestors. This, of course, increases the dimension of the matrix by a factor of $m + 1$. A replica of type a, b results in two replicas, one of type $a + 1, b$ and one of type $0, b + 1$ unless $a = n$ or $b = m$, in which case a, b goes into a, b. For $n = 2$ and $m = 3$, one calculates

$$F = \begin{pmatrix} 0&1&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

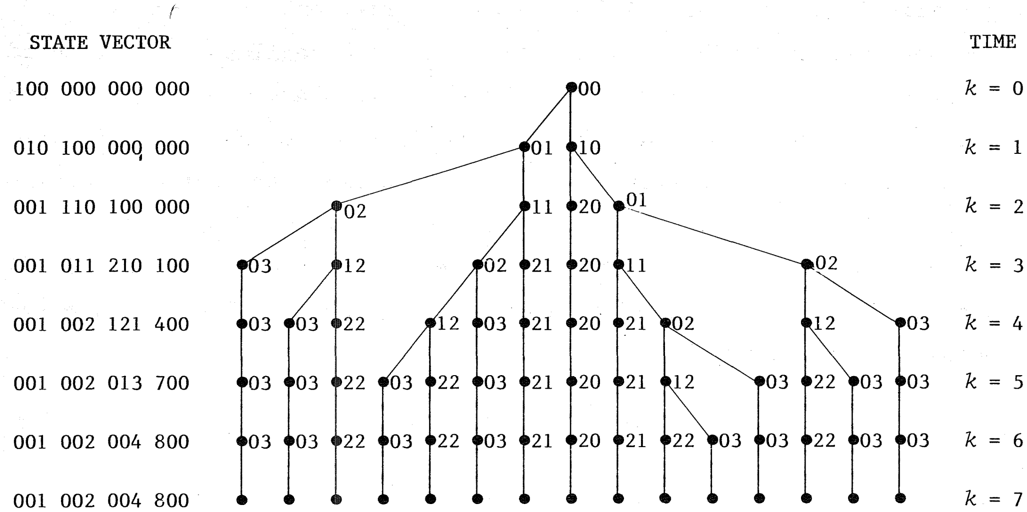

FIGURE 5. Cumulative Diagram

The diagram has a lack of symmetry which cannot be helped; it does, however, place the final replicas equidistant on a straight line and does not move them after they are first placed. The strategy for positioning the replicas is another problem and one we are not going to address. We would like to point out that the matrix is in a form

$$\begin{pmatrix} A & B & 0 & 0 \\ 0 & A & B & 0 \\ 0 & 0 & A & B \\ 0 & 0 & 0 & I \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The A matrix is due to the renaming of existing replicas, while the B matrix is due to replication. The obvious extension of this observation is useful in both setting up the F matrix and its subsequent calculations. Since for Option C there is a limit to the number of generations as well as offsprings, the state vector must eventually be constant. So, for some k and for all integers greater, $\mathbf{f}(0)F^k = \mathbf{f}(0)F^{k+1}$. The minimal such k^* is mn and, further, we note that the sum of the $\mathbf{f}(k)$ coordinates is given by equation (7) as is the sum of the first row of F^{k^*}, since $\mathbf{f}(0) = (1, 0, \ldots, 0)$.

Using the definitions, one can write relationships where complete tables can be generated to show various totals at any time. For n replicas per primary, $s_{m,k}$ denotes the number of replicas produced in the kth time frame under the m generation restriction and $S_{m,k}$ the cumulative number. Similarly, $p_{m,k}$ refers to those coming into production during the kth time frame and $P_{m,k}$ the cumulative number (see Table 1):

$$s_{m,k} = \sum_{i=1}^{n} s_{m-1,k-i}$$

$$s_{m,k+1} = S_{m,k} - P_{m,k}$$

$$p_{m,k} = 2s_{m,k} - s_{m,k+1}$$

$$P_{m,k} = \sum_{i-1}^{n} P_{m-1,k-i}$$

$$P_{m,k} = 2S_{m,k} - S_{m,k+1}$$

For Option B, a replica begins production when it has completed its n replications. Therefore, $p_k = s_{k-n}$ for k less than cutoff; at cutoff, the remaining replicas begin production. Finally, in Option A, since replication is simultaneous, $p_k = s_{k-1}$.

TABLE 1

Results for four replicas per primary with m = 2 and m = 3

k	$s_{2,k}$	$p_{2,k}$	$S_{2,k}$	$P_{2,k}$	$s_{3,k}$	$p_{3,k}$	$S_{3,k}$	$P_{3,k}$
0	1	0	1	0	1	0	1	0
1	1	0	2	0	1	0	2	0
2	2	1	4	1	2	0	4	0
3	3	2	7	3	4	1	8	1
4	4	4	11	7	7	4	15	5
5	4	5	15	12	10	7	25	12
6	3	4	18	16	13	12	38	24
7	2	3	20	19	14	15	52	39
8	1	2	21	21	13	16	65	55
9	0	0	21	21	10	14	75	69
10	0	0	21	21	6	9	81	78
11	0	0	21	21	3	5	84	83
12	0	0	21	21	1	2	85	85
13	0	0	21	21	0	0	85	85
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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