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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be submitted on a separate signed sheet, or sheets. Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-496 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

Show that the centroid of the triangle whose vertices have coordinates (F_n, L_n), (F_{n+1}, L_{n+1}), (F_{n+6}, L_{n+6}) is (F_{n+4}, L_{n+4}).

B-497 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

For d an odd positive integer, find the area of the triangle with vertices (F_n, L_n), (F_{n+d}, L_{n+d}), and (F_{n+2d}, L_{n+2d}).

B-498 *Proposed by Herta T. Freitag, Roanoke, VA*

Characterize the positive integers k such that, for all positive integers n, $F_n + F_{n+k} \equiv F_{n+2k} \pmod{10}$.

B-499 *Proposed by Herta T. Freitag, Roanoke, VA*

Do the Lucas numbers analogue of B-498.

B-500 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $A(n)$ and $B(n)$ be polynomials of positive degree with integer coefficients such that $B(k) | A(k)$ for all integers k. Must there exist a nonzero integer h and a polynomial $C(n)$ with integer coefficients such that $hA(n) = B(n)C(n)$?

B-501 *Proposed by J. O. Shallit & J. P. Yamron, U.C., Berkeley, CA*

Let α be the mapping that sends a sequence $X = (x_1, x_2, \ldots, x_{2k})$ of length $2k$ to the sequence of length k

$$\alpha(X) = (x_1x_{2k}, x_2x_{2k-1}, x_3x_{2k-2}, \ldots, x_kx_{k+1}).$$

Let $V = (1, 2, 3, \ldots, 2^h)$, $\alpha^2(V) = \alpha(\alpha(V))$, $\alpha^3(V) = \alpha(\alpha^2(V))$, etc. Prove that $\alpha(V), \alpha^2(V), \ldots, \alpha^{h-1}(V)$ are all strictly increasing sequences.

SOLUTIONS

Where To Find Perfect Numbers

B-472 *Proposed by Gerald E.Bergum, S. Dakota State Univ., Brookings, SD*

Find a sequence $\{T_n\}$ satisfying a second-order linear homogeneous recurrence $T_n = aT_{n-1} + bT_{n-2}$ such that every even perfect number is a term in $\{T_n\}$.

Solution by Graham Lord, Université Laval, Québec

A (trivial) solution to this problem is the sequence of even integers $a = 2$ and $b = -1$, with seeds $T_1 = 2$ and $T_2 = 4$. With $a = 6$ and $b = -8$, the sequence T_n is $2^{n-1}(2^n - 1)$ if $T_1 = 1$ and $T_2 = 6$. The proof is immediate:

$$\begin{aligned} T_n &= 6T_{n-1} - 8T_{n-2} \\ &= 6(2^{2n-3} - 2^{n-2}) - 8(2^{2n-5} - 2^{n-3}) \\ &= 2^{2n-1} - 2^{n-1}. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Edgar Krogt, Bob Prielipp, Sahib Singh, Paul Smith, J. Suck, Gregory Wulczyn, and the proposer.

Primitive Fifth Roots of Unity

B-473 *Proposed by Philip L. Mana, Albuquerque, NM*

Let

$$a = L_{1000},\ b = L_{1001},\ c = L_{1002},\ d = L_{1003}.$$

Is $1 + x + x^2 + x^3 + x^4$ a factor of $1 + x^a + x^b + x^c + x^d$? Explain.

Solution by Paul S. Bruckman, Carmichael, CA

It is easy to verify that $\{L_n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 4. Specifically,

$$L_{4k} \equiv L_0 = 2,\ L_{4k+1} \equiv L_1 = 1,\ L_{4k+2} \equiv L_2 = 3$$

and

$$L_{4k+3} \equiv L_3 = 4 \pmod 5,\ k = 0, 1, 2, \ldots .$$

Therefore, $a \equiv 2$, $b \equiv 1$, $c \equiv 3$, and $d \equiv 4 \pmod 5$.

A polynomial $p(x)$ divides another polynomial $q(x)$ if $q(x_0) = 0$ for all x_0 such that $p(x_0) = 0$. Letting $p(x) = 1 + x + x^2 + x^3 + x^4$, we see that $p(x)$ is the cyclotomic polynomial $(x^5 - 1)/(x - 1)$, which has four complex zeros equal to the complex fifth roots of unity. Let θ denote any of these roots. Since $p(\theta) = 0$, it suffices to show that $q(\theta) = 0$, where $q(x) \equiv 1 + x^a + x^b + x^c + x^d$.

Now $\theta^5 = 1$, and it follows from this and the congruences satisfied by a, b, c, and d, that

$$q(\theta) = 1 + \theta^2 + \theta + \theta^3 + \theta^4 = p(\theta) = 0.$$

This shows that the answer to the problem is affirmative.

Also solved by C. Georghiou, Walther Janous, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Sequence of Congruences

B-474 *Proposed by Philip L. Mana, Albuquerque, NM*

Are there an infinite number of positive integers n such that

$$L_n + 1 \equiv 0 \pmod{2n}?$$

Explain.

Solution by Bob Prielipp, Univ. of Wisconsin-Oshkosh, WI

Induction will be used to show that

$$L_{2^k} + 1 \equiv 0 \pmod{2^{k+1}}$$

for each nonnegative integer k. Clearly, the desired result holds when $k = 0$ and when $k = 1$. Assume that

$$L_{2^j} + 1 \equiv 0 \pmod{2^{j+1}},$$

where j is an arbitrary positive integer. Then

$$L_{2^j} = q \cdot 2^{j+1} - 1$$

for some integer q. It is known that if m is even, $L_m^2 = L_{2m} + 2$ [see p. 189 of "Divisibility and Congruence Relations" by Verner E. Hoggatt, Jr. and Gerald E. Bergum in the April 1974 issue of this journal]. Thus,

$$\begin{aligned} L_{2^j} + 1 = L_{2(2^j)} + 1 &= (L_{2^j})^2 - 2 + 1 \\ &= (q \cdot 2^{j+1} - 1)^2 - 1 \\ &= (q^2 \cdot 2^{2j+2} - q \cdot 2^{j+2}) + (1 - 1) \\ &\equiv 0 \pmod{2^{j+2}}. \end{aligned}$$

Also solved by Paul S. Bruckman, C. Georghiou, Graham Lord, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Wrong Sign

B-475 *Proposed by Herta T. Freitag, Roanoke, VA*

The problem should read: "Prove that $|S_3(n)| - S_1^2(n)$ is $2[(n + 1)/2]$ times a triangular number."

Solution by Paul Smith, Univ. of Victoria, B.C., Canada

It is easily shown that if $n = 2m$,

(i) $S_3(n) = -m^2(4m + 3)$

(ii) $S_1^2(n) = m^2$

(iii) $2[(n + 1)/2] = 2m$.

Thus

$$|S_3(n)| - S_1^2(n) = m^2(4m + 2) = 2m \cdot \frac{2m(2m + 1)}{2} = 2[(n + 1)/2] \cdot T_n.$$

If $n = 2m + 1$,

$$S_3(n) = -m^2(4m + 3) + (2m + 1)^3,$$

$$S_1^2(n) = (m + 1)^2$$

and

$$2[(n + 1)/2] = 2(m + 1).$$

And now

$$|S_3(n)| - S_1(n)^2 = 2(2m^3 + 5m^2 + 4m + 1) = 2(m + 1)(2m + 1)(m + 1)$$

(continued)

$$= 2(m + 1) \cdot \frac{(2m + 1)(2m + 2)}{2} = 2[(n + 1)/2] \cdot T_n.$$

Also solved by Paul S. Bruckman, Graham Lord, Bob Prielipp, Sahib Singh, J. Suck, Gregory Wulczyn, and the proposer.

Multiples of Triangular Numbers

B-476 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k.$$

Prove that $|S_4(n) + S_2(n)|$ is twice the square of a triangular number.

Solution by Graham Lord, Université Laval, Québec

As $(k + 1)^4 - k^4 + (k + 1)^2 - k^2 = 2(k + 1)^3 + 2k^3$, then

$$\begin{aligned} S_4(2m) + S_2(2m) &= -2(1^3 + 2^3 + \cdots + (2m)^3) \\ &= -2\{2m(2m + 1)/2\}^2. \end{aligned}$$

And

$$\begin{aligned} S_4(2m + 1) + S_2(2m + 1) &= S_4(2m) + S_2(2m) + (2m + 1)^4 + (2m + 1)^2 \\ &= 2\{(2m + 1)(2m + 2)/2\}^2. \end{aligned}$$

Also solved by Paul S. Bruckman, Walther Janous, H. Klauser, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, Gregory Wulczyn, and the proposer.

Telescoping Series

B-477 *Proposed by Paul S. Bruckman, Sacramento, CA*

Prove that

$$\sum_{n=2}^{\infty} \text{Arctan} \frac{(-1)^n}{F_{2n}} = \frac{1}{2} \text{Arctan} \frac{1}{2}.$$

Solution by C. Georghiou, Univ. of Patras, Patras, Greece

It is known [see, e.g., Theorem 5 of "A Primer for the Fibonacci Numbers—Part IV" by V. E. Hoggatt, Jr. and I. D. Ruggles, this Quarterly, Vol. 1, no. 4 (1963):71] that

$$\sum_{m=1}^{\infty} (-1)^{m+1} \text{Arctan} \frac{1}{F_{2m}} = \text{Arctan} \frac{\sqrt{5} - 1}{2}.$$

The problem is readily solved by noting that

$$(-1)^m \text{Arctan } x = \text{Arctan}(-1)^m x$$

and that

$$\text{Arctan } 1 - \text{Arctan } \frac{\sqrt{5}-1}{2} = \frac{1}{2} \text{ Arctan } \frac{1}{2}.$$

Also solved by John Spraggon, J. Suck, and the proposer.

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