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A VARIANT OF NIM AND A FUNCTION DEFINED BY FIBONACCI REPRESENTATION

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The game Spite Nim I was introduced by Jesse Croach in [2] and discussed further briefly in [3]. No solution was given for the same in these references, and some questions were raised about a partial solution for certain simple cases of the game. This note will solve part of one of these questions, and will show that the solution is closely related to the golden ratio $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Spite Nim is played in the following way: Two players pick from several rows of counters. On a player's turn to move, he announces a positive number of counters. This number must be less than or equal to the number of counters in the longest row. His opponent then indicates from which row these counters are to be taken. (This is the "spite" option.) This row must have at least as many counters as the call. The players alternate moves. The player who takes the last counter wins.

In this note only the case of two rows will be considered. A configuration of two rows of lengths n and r will be denoted by (n, r). This actually should be considered an unordered pair.

Given any pair, a person receiving such a pair can either make a call which with best play on both sides will give him a win, or he loses, no matter what call he makes. In the first case, the position is called unsafe (it is unsafe to leave it to your opponent); in the second case, it is called safe.

It will be shown that for each n there is an $r \leqslant n$ for which (r, n) is safe, and if $s < r$, (n, s) is unsafe. The number r will be shown to be equal to a function of n which has been previously studied.

Define a function f on the natural numbers by $f(1) = 1$, and for $n > 1$, $f(n) = r$, where r is the smallest number for which $r + f(r) \geqslant n$. Since $f(r) \geqslant 1$, such an r clearly exists.

Theorem 1

For all natural numbers n, $(n, f(n))$ is safe, and if $s < f(n)$, (n, s) is unsafe.

Proof: Use induction. $(1, 0)$ is clearly unsafe, while $(1, 1)$ is safe. So the theorem holds for $n = 1$.

Assume Theorem 1 holds for all $s < n$. Then, first, if $s < f(n)$, then (n, s) is unsafe.

To show this, suppose a player is given (n, s). Since $s < f(n)$, $s + f(s) < n$, by definition of f. Therefore, $n - f(s) > s$. So, on the call $n - f(s)$, the resulting pair is $(f(s), s)$, which by hypothesis is safe.

Secondly, $(n, f(n))$ is safe. On a call of $r \leqslant f(n)$, take from the second row to get $(n, f(n) - r)$. This has just been shown to be unsafe.

On a call $r > f(n)$, the result is $(n - r, f(n))$. But since $n \leqslant f(n) + f(f(n))$, $n - r < f(f(n))$. So by hypothesis, $(n - r, f(n))$ is unsafe; thus, Theorem 1 is proved.

Now, reexamine f. $f(n)$ is in fact the same as $e(n)$, defined in [1].

To show $f(n) = e(n)$, we will show $e(n)$ satisfies the recursion $f(n)$ does. Since $f(1) = 1 = e(1)$, this will show the functions are identical.

First, write n in Fibonacci notation. Let F_m be the mth Fibonacci number. Then $n = F_{r_1} + F_{r_2} + \cdots + F_{r_k}$, where $r_i - r_{i+1} \geqslant 2$, and $r_k \geqslant 2$. By definition, $e(n) = F_{r_1 - 1} + F_{r_2 - 1} + \cdots + F_{r_k - 1}$.

If $r_k \neq 2$, then

$$e(e(n)) = F_{r_1 - 2} + F_{r_2 - 2} + \cdots + F_{r_k - 2}.$$

So $e(n) + e(e(n)) = n$. Also, since $e(n)$ is nondecreasing, if $s < e(n)$, then $s + e(s) < e(n) + e(e(n)) = n$. So $e(n)$ satisfies the recursion here.

If $r_k = 2$, again $e(n) = F_{r_1 - 1} + \cdots + F_{r_k - 1}$. However, since $r_k - 1 = 1$, this no longer expresses $e(n)$ in correct Fibonacci representation, so the preceding argument requires modification. We can say, however, that $e(n) > F_{r_1 - 1} + \cdots + F_{r_{k-1} - 1}$, so $e(e(n)) \geqslant F_{r_1 - 2} + \cdots + F_{r_{k-1} - 2}$, and $e(n) + e(e(n)) \geqslant n$.

Also, if $s < e(n)$, then $s \leqslant F_{r_1 - 1} + \cdots + F_{r_{k-1} - 1}$. Thus

$$e(s) + e(e(s)) \leqslant F_{r_1} + \cdots + F_{r_{k-1}} \leqslant n - 1 < n.$$

So here, also, $e(n)$ satisfies the recursion. Therefore, $e(n) = f(n)$ for all natural numbers n.

Alternate formulas for $e(n)$ are given in [1] that indicate how close $e(n)$ is to $\alpha^{-1}n$. Let $\{x\}$ be the integer nearest x, and let $[x]$ be the greatest integer $< x$. Then if $n = F_{r_1} + \cdots + F_{r_k}$ is the Fibonacci representation for n,

$$e(n) = \{\alpha^{-1}n\} \qquad \text{if } r_k \neq 2,$$

$$e(n) = [\alpha^{-1}n] + 1 \quad \text{if } r_k = 2.$$

Deeper inspection of Fibonacci notation might possibly solve the two-row game, but I have been unable to do so.

To close this note, here are two weak results regarding safe (n, s) with $s > e(n)$.

Theorem 2

Exactly one of the pairs $(n, e(n)+1)$ and $(n-1, e(n)+1)$ is safe.

Proof: If $(n, e(n)+1)$ is unsafe, the only call must be 1. But then, $(n-1, e(n)+1)$ must be safe. The converse follows in the same way.

Consider for any natural number n the number

$$h(n) = \#\{s : s \leqslant n,\ (s, e(s)+1) \text{ is safe}\}.$$

Since $e(s) = e(s-1)$ for approximately $(1-\alpha^{-1})n$ numbers $s \leqslant n$, this gives, with Theorem 2,

$$\frac{1}{2}(1-\alpha^{-1}) \leqslant \frac{h(n)}{n} \leqslant \frac{1}{2}(n-\alpha^{-1}) + \alpha^{-1}.$$

Theorem 3

If $e(n) < s < n$, (n, s) is unsafe, and r is a winning call, then $(n, n-r)$ is unsafe and $n-s$ is a winning call.

Proof: If (n, s) is unsafe and r is a winning call, then $(n-r, s)$ and $(n, s-r)$ are both safe. But the call $n-s$ on $(n, n-r)$ gives rise to the identical results.

Perhaps these may help determine for what $s > e(n)$ is (n, s) safe. Results for the three- or more-row game would also be interesting.

REFERENCES

1. L. Carlitz. "Fibonacci Representation." *The Fibonacci Quarterly* 1, no. 1 (1963):57-63.
2. Jesse Croach. "Spite Nim I." Problem 371. *Journal of Recreational Mathematics* 8, no. 1 (1975):47.
3. *Journal of Recreational Mathematics* 11, no. 1 (1978-1979):48.

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