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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or any other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-356 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Consider a set of r types of letter with n_i occurrences of letter i. How many words can we form, using some or all of these letters?

If we use k_i of letter i, then there are obviously $\binom{\Sigma k_i}{k_1, \ldots, k_r}$ ways to form a word, and the desired number is $\sum_{k_i \leq n_i} \binom{\Sigma k_i}{k_1, \ldots, k_r}$. When $r = 2$, this can be readily evaluated using properties of Pascal's triangle and we get $\binom{n_1 + n_2 + 2}{n_1 + 1} - 1$. W. O. J. Moser has found a nice combinatorial derivation of this result, but neither approach works for $r > 2$.

Moser's solution for $r = 2$ is as follows: In the case $r = 2$,

$$\sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \binom{i + j}{i} \tag{**}$$

is the number of ways of forming words with some of m A's and n B's. Any such word with i A's and j B's can be extended to a word of $m + 1$ A's and $n + 1$ B's by appending $m + 1 - i$ A's and $n + 1 - j$ B's to it. If our original word begins with an A, we append a block of $m + 1 - i$ A's followed by a block of $n + 1 - j$ B's at the beginning. If the original word begins with a B, we append the block of B's followed by the block of A's at the beginning. The empty word can be extended in two ways: AA ... ABB ... A or BB ... BAA ... A. Otherwise, we have a one-to one correspondence between our original words and words formed from all of $m + 1$ A's and $n + 1$ B's. The reverse correspondence is to take any word of $m + 1$ A's and $n + 1$ B's and delete its first two blocks (i.e., constant subintervals). Since the empty word arises from two extended words, we have $\binom{m+n+2}{m+1} - 1$ of our original words.

As an illustration, let $m = n = 1$.

Original Word	Extended Word
-	AABB or BBAA
A	ABBA
B	BAAB
AB	ABAB

H-357 *Proposed by Clark Kimberling, Univ. of Evansville, Evansville, IN*

For any positive integer N, arrange the fractional parts of the first N integral multiples of $\alpha = (1 + \sqrt{5})/2$ in increasing order:

$$\{k_1\alpha\} < \{k_2\alpha\} < \cdots < \{k_N\alpha\}.$$

Is $k_n + k_{N+1-n}$ a sum of two Fibonacci numbers for $n = 1, 2, 3, \ldots, N$?

I have not been able to prove that $k_n + k_{N+1-n}$ is always a sum of two Fibonacci numbers. However, a computer has verified that it is so for $N = 1, 2, \ldots, 666$.

The following table may be helpful:

N	αN	$\{\alpha N\}$	$k_1, k_2, \ldots, k_N$	$k_1 + k_N, k_2 + k_{N-2}, \ldots, k_N + k_1$
1	1.618	.618	1	2
2	3.236	.236	2 1	3 3
3	4.854	.854	2 1 3	5 2 5
4	6.472	.472	2 4 1 3	5 5 5 5
5	8.090	.090	5 2 4 1 3	8 3 8 3 8
6	9.708	.708	5 2 4 1 6 3	8 8 5 5 8 8
7	11.326	.326	5 2 7 4 1 6 3	8 8 8 8 8 8 8
8	12.944	.944	5 2 7 4 1 6 3 8	13 5 13 5 5 13 5 13
9	13.562	.562	5 2 7 4 9 1 6 3 8	13 5 13 5 18 5 13 5 13

As you see, all numbers in the fifth column are sums of two Fibonacci numbers. For $N = 662$, for example, there are six (and only six) different numbers $k_n + k_{N+1-n}$ as n ranges from 1 to 662; they are:

$$\begin{aligned} 144 &= 89 + 55 \\ 377 &= 233 + 144 \\ 521 &= 377 + 144 \\ 754 &= 377 + 377 \\ 987 &= 610 + 377 \\ 1131 &= 987 + 144 \end{aligned}$$

H-358 *Proposed by Andreas N. Philippou, Univ. of Patras, Patras, Greece*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{n_1,\ldots,n_k} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1}, \; n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$. Show that

$$\sum_{n=0}^{\infty} (f_{n+1,r}^{(k)}/2^n) = 2^{rk}.$$

You may note that the present problem reduces to H-322(c) for $r = 1$ (and $k \geqslant 2$), because of Theorem 2.1 of Philippou and Muwafi [1]. In addition, the present problem includes as special cases [for $k = 1$, $r = 1$, and $k = 1$, $r(\geqslant 1)$] the following infinite sums; namely,

$$\sum_{n=0}^{\infty} (1/2^n) = 2 \quad \text{and} \quad \sum_{n=0}^{\infty} \left[\binom{n + r - 1}{n}/2^n\right] = 2^r.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the kth Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

H-359 *Proposed by Paul S. Bruckman, Carmichael, CA*

Define the "Zetanacci" numbers $Z(n)$ as follows:

$$Z(n) = \prod_{p^e \| n} F_{e+1}, \; n = 1, 2, 3, \ldots \text{ [with } Z(1) = 1]. \tag{1}$$

For example, $Z(n) = 1$, $n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, \ldots$; $Z(n) = 2$, $n = 4, 9, 12, 18, 20, \ldots$; $Z(8) = 3$, $Z(16) = 5$, $Z(135{,}000) = Z(2^3 3^3 5^4) = 45$, etc.

(A) Show that the (Dirichlet) generating function of the Zetanacci numbers is given by:

$$\sum_{n=1}^{\infty} Z(n)n^{-s} = \prod_p (1 - p^{-s} - p^{-2s})^{-1}, \tag{2}$$

the product taken over all primes.

(B) Show that

$$\prod_p (1 - p^{-s} - p^{-2s}) = \sum_{n=1}^{\infty} \mu(P(n)) \cdot |\mu(n/P(n))| \cdot n^{-s},$$

where μ is the Möbius function and

$$P(n) = \prod_{p|n} p \text{ [with } P(1) = 1].$$

SOLUTIONS

Rational Thirds

H-339 *Proposed by Charles R. Wall, Trident Technical College, Charleston, CA (Vol. 20, No. 2, May 1982)*

A dyadic rational is a proper fraction whose denominator is a power of 2. Prove that 1/4 and 3/4 are the *only* dyadic rationals in the classical Cantor ternary set of numbers representable in base three using only 0 and 2 as digits.

Solution by the proposer

Clearly $1/2 = .\overline{1}$ (base three) is not in the set, but $1/4 = .\overline{02}$ and $3/4 = .\overline{20}$ are. The other cases require a lemma:

If $k \geqslant 3$ and $0 \leqslant a < 2^{k-2}$, the numbers $\pm 3^a$ are distinct modulo 2^k.

This assertion is true for $k = 3$ by observation: $3^0 \equiv 1$, $-3^0 \equiv 7$, $3^1 \equiv 3$, and $-3^1 \equiv 5$ (all mod 8). Thus, we may assume $k \geqslant 4$. That the numbers 3^a are distinct (mod 2^k) rests on the congruence

$$3^{2^{k-3}} \equiv 1 + 2^{k-1} \pmod{2^k},$$

which is easily proved by induction for $k \geqslant 4$, and its corollary

$$3^{2^{k-2}} \equiv 1 \pmod{2^k}.$$

To show that the numbers 3^a are distinct from their negatives, note that $3^x \equiv (-1)^x$ (mod 4). If $k \geqslant 4$ and $0 \leqslant b < a < 2^{k-2}$ and $3^a \equiv -3^b$ (mod 2^k), then $3^{a-b} \equiv -1$ (mod 2^k), so $a - b$ is odd. Then $3^{2(a-b)} \equiv 1$ (mod 2^k), so 2^{k-2} divides $2(a-b)$, and thus 2^{k-3} divides the odd number $a - b$, which is impossible if k 4.

Let $f(t)$ be the fractional part of t: $f(t) = t - [t]$, where the brackets denote the greatest integer function. For $k \geqslant 3$, by the lemma, each dyadic rational with denominator 2^k can be written uniquely as $f(\pm 3^a/2^k)$, $0 \leqslant a < 2^{k-2}$. If a fraction $x = f(\pm 3^a/2^k)$ is in the Cantor set, so (by shifting the ternary point) is $f(3x) = f(\pm 3^{a+1}/2^k)$, and so is the 2's complement $1 - x = f(\mp 3^a/2^k)$. Thus, if any dyadic rational $x = f(\pm 3^a/2^k)$ is in the set, all such fractions with the same denominator are. However, the two fractions closest to 1/2 are forbidden, so all are.

Also solved by P. Bruckman.

Making a Difference

H-340 *Proposed by Verner E. Hoggatt, Jr. (deceased) (Vol. 20, No. 2, May 1982)*

Let $A_2 = B$, $A_4 = C$, and $A_{2n+4} = A_{2n} - A_{2n+2}$ $(n = 1, 2, 3, \ldots)$. Show:

a. $A_{2n} = (-1)^{n+1}(F_{n-2}B - F_{n-1}C)$

b. If $A_{2n} > 0$ for all $n > 0$, then $B/C = (1 + \sqrt{5})/2$

c. $A_{2n} = C^{n-1}/B^{n-2}$

Solution by Paul Bruckman, Carmichael, CA

For all $n \geqslant 1$, let

$$G_n = A_{2n}. \tag{1}$$

The given recursion is then transformed to the following recursion:

$$G_{n+2} + G_{n+1} - G_n = 0,\ n = 1, 2, 3, \ldots, \tag{2}$$

with initial conditions

$$G_1 = B,\ G_2 = C. \tag{3}$$

The characteristic polynomial $p(z)$ of (2) is given by

$$p(z) = z^2 + z - 1 = (z + \alpha)(z + \beta), \tag{4}$$

where α and β are the usual Fibonacci constants. Hence, there exist constants p and q such that, for all n,

$$G_n = p(-\alpha)^n + q(-\beta)^n. \tag{5}$$

We find p and q by setting $n = 1$ and $n = 2$ in (5) and using (3). After simplification, we find the following expression (which is readily verifiable):

$$G_n = (-1)^{n+1}(F_{n-2}B - F_{n-1}C),\ n = 1, 2, 3, \ldots . \tag{6}$$

Note that the expression in (6) is of the same form as given in (5), and moreover satisfies (3). Hence, A_{2n} is given by (6).

Thus,

$$G_{2n} = F_{2n-1}C - F_{2n-2}B \text{ for } n \geqslant 1$$

and

$$G_{2n+1} = F_{2n-1}B - F_{2n}C \quad \text{for } n \geqslant 0.$$

Since $G_n > 0$ for all $n > 0$, we have $B > C > 0$ and

$$F_{2n}/F_{2n-1} < B/C < F_{2n-1}/F_{2n-2},\ n = 2, 3, 4, \ldots . \tag{7}$$

Taking limits in (7) as $n \to \infty$, each extreme expression approaches α, which implies $B/C = \alpha$. Q.E.D.

Also solved by H. Freitag, C. Georghiou, W. Janous, G. Lord, A. Shannon, and the proposer.

Late Acknowledgment: *G. Wulczyn solved* H-332.

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