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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-360</u> *Proposed by M. Wachtel, Zürich, Switzerland*

Let

$$F_nF_{n+1} + F_{n+2}^2 = A_1$$
$$F_{n+1}F_{n+2} + F_{n+3}^2 = A_2$$
$$F_{n+2}F_{n+3} + F_{n+4}^2 = A_3$$

Show that:

(1) No integral divisor of A is congruent to 3 or 7 modulo 10.
(2) $A_1A_2 + 1$, as well as $A_1A_3 + 1$, are products of two consecutive integers.

<u>H-361</u> *Proposed by Verner E. Hoggatt, Jr. (deceased)*

Let $H_n = P_{2n}/2$, $n > 0$, where P_n denotes the nth Pell number. Show that

$$H_m + H_n \neq P_k$$
$$H_m + H_n = P_k + P_{k-1}$$

if and only if $m = n + 1$, where $k = 2n + 1$, and

$$P_{2n+2}/2 + P_{2n}/2 = ((2P_{2n+1} + P_{2n}) + P_{2n})/2 = P_{2n+1} + P_{2n}.$$

<u>Editorial Note</u>: *Refer to the January 1972 article on Generalized Zeckendorf Theorem for Pell Numbers.*

<u>H-362</u> *Proposed by Stanley Rabinowitz, Merrimack, NH*

Let Z be the ring of integers modulo n. A Lucas Number in this ring is a member of the sequence $\{L_k\}$, $k = 0, 1, 2, \ldots$, where $L_0 = 2$, $L_1 = 1$, and $L_{k+2} \equiv L_{k+1} + L_k$ for $k \geqslant 0$. Prove that, for $n > 14$, all members of Z_n are Lucas numbers if and only if n is a power of 3.

Remark: A similar, but more complicated, result is known for Fibonacci numbers. See [1]. I do not have a proof of the above proposal, but I suspect a proof similar to the result in [1] is possible; however, it should be considerably simpler, because there is only one case to consider rather than seven cases.

To verify the conjecture, I ran a computer program that examined Z_n for all n between 2 and 10000 and found that the only cases where all members of Z_n were Lucas numbers were powers of 3, and the exceptional values $n = 2, 4, 6, 7$, and 14 (the same exceptions found in [1]). This is strong evidence for the truth of the conjecture.

Reference

1. S. A. Burr. "On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues." *The Fibonacci Quarterly* 9 (1971):497.

H-363 *Proposed by Andreas N. Philippou, University of Patras, Greece*

For each fixed integer $k \geqslant 2$, let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, i.e., $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)}, & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)}, & \text{if } n \geqslant k+1. \end{cases}$$

Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{f_{m^n}^{(k)}} \quad (k \geqslant 2,\ m \geqslant 2).$$

Remark: The Fibonacci sequence of order k appears in the work of Philippou and Muwafi, *The Fibonacci Quarterly* 20 (1982);28-32.

H-364 *Proposed by M. Wachtel, Zürich, Switzerland*

For every n, show that no integral divisor of L_{2n+1} is congruent to 3 or 7, modulo 10.

SOLUTIONS

The Root of the Problem

H-341 *Proposed by Paul S. Bruckman, Concord, CA*
(Vol. 20, No. 2, May 1982)

Find the real roots, in exact radicals, of the polynomial equation

$$p(x) \equiv x^6 - 4x^5 + 7x^4 - 9x^3 + 7x^2 - 4x + 1 = 0. \tag{1}$$

Solution by the proposer

We note that $p(0) \neq 0$ and $p(x) = x^6 p(1/x)$. Let

$$y = x + x^{-1}. \tag{2}$$

Then $y^2 = x^2 + x^{-2} + 2$ and $y^3 = x^3 + x^{-3} + 3y$; hence,

$$x^{-3}p(x) = x^3 + x^{-3} - 4(x^2 + x^{-2}) + 7(x + x^{-1}) - 9$$

$$= y^3 - 3y - 4(y^2 - 2) + 7y - 9,$$

or

$$y^3 - 4y^2 + 4y - 1 = 0. \tag{3}$$

This polynomial in y may be readily factored, noting that it vanishes for $y = 1$. Thus,

$$(y - 1)(y^2 - 3y + 1) = (y - 1)(y - a^2)(y - b^2) = 0.$$

Now, we may solve for x in terms of y, first multiplying (2) throughout by $x : x^2 - xy + 1 = 0$, or

$$x = \tfrac{1}{2}(y \pm \sqrt{y^2 - 4}). \tag{4}$$

Setting $y = 1$ or $y = b^2$ in (4) yields imaginary roots of (1) (and, moreover, of unit modulus). Setting $y = a^2$, however, yields real roots, which after a little manipulation are found to be as follows:

$$x_1 = \tfrac{1}{4}(3 + \sqrt{5} + \sqrt{6\sqrt{5} - 2}) \doteq 2.1537214, \tag{5}$$

$$x_2 = \tfrac{1}{4}(3 + \sqrt{5} - \sqrt{6\sqrt{5} - 2}) \doteq .46431261 \doteq 1/x_1. \tag{6}$$

Also solved by W. Blumberg, H. Freitag, W. Janous, D. Laurie, D. Russell, C. Shields, and M. Wachtel.

Say A

H-342 *Proposed by Paul S. Bruckman, Corcord, CA*
(Vol. 20, No. 3, August 1982)

Let

$$A_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n - 2k}{n} 4^k, \quad n = 0, 1, 2, \ldots . \tag{1}$$

Prove that

$$\sum_{k=0}^{n} A_k A_{n-k} = 4^n F_{n+1}. \tag{2}$$

Solution by the proposer

Proof #1: The well-known Legendre polynomials are defined by the generating function

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n \text{ (valid for } |x| < 1,\ |z| < 1), \tag{3}$$

and are given explicitly as

$$P_n(x) = 2^{-n} \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n - 2k}{n}(-1)^k x^{n-2k}. \tag{4}$$

(see, for example, formulas 22.3.8 and 22.9.12 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ed. Milton Abramowitz & Irene A. Stegun, National Bureau of Standards Applied Mathematics Series 55, issued June 1964, 9th printing, November 1970, with corrections).

In (3) and (4), set $x = \frac{1}{2}i$ and replace z in (3) by $-iz$. Then

$$(1 - z - z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(\tfrac{1}{2}i)(-iz)^n, \tag{5}$$

and, using the definition of A_n in (1):

$$P_n(\tfrac{1}{2}i) = (\tfrac{1}{4}i)^n A_n. \tag{6}$$

Thus,

$$(1 - z - z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n(\tfrac{1}{4}z)^n. \tag{7}$$

Squaring both sides of (7), we obtain the generating function of the Fibonacci numbers:

$$(1 - z - z^2)^{-1} = \sum_{n=0}^{\infty} F_{n+1}z^n = \sum_{n=0}^{\infty} (\tfrac{1}{4}z)^n \sum_{k=0}^{n} A_k A_{n-k}$$

(the last result by convolution). We obtain (2) by comparison of coefficients in the last two expressions. Q.E.D.

The following is a more direct proof of the foregoing result.

Proof #2: Let

$$f(z) \equiv \sum_{n=0}^{\infty} A_n(\tfrac{1}{4}z)^n. \tag{8}$$

Then

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} (\tfrac{1}{4}z)^n \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n-2k}{n}4^k = \sum_{n,k=0}^{\infty} (\tfrac{1}{4}z)^{n+2k}\binom{n+2k}{k}\binom{2n+2k}{n+2k}4^k \\
&= \sum_{n,k=0}^{\infty} (\tfrac{1}{4})^{n+k} z^{n+2k}\binom{2n+2k}{n+k}\binom{n+k}{k} \\
&= \sum_{n,k=0}^{\infty} (\tfrac{1}{4}z^2)^k \binom{-\frac{1}{2}}{n+k}\binom{n+k}{k}(\tfrac{1}{4}z)^n(-4)^{n+k} \\
&= \sum_{n,k=0}^{\infty} (-z)^n(-z^2)^k \binom{-\frac{1}{2}}{n}\binom{-\frac{1}{2}-n}{k} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-z)^n(1-z^2)^{-\frac{1}{2}-n} \\
&= (1-z^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-z)^n(1-z^2)^{-n} = (1-z^2)^{-\frac{1}{2}}\left\{1 - \frac{z}{1-z^2}\right\}^{-\frac{1}{2}},
\end{aligned}$$

or

$$f(z) = (1 - z - z^2)^{-\frac{1}{2}}. \tag{9}$$

The rest of the proof now proceeds as in the first proof, after (7). Q.E.D.

The first few values of $(A_n)_{n=0}^{\infty}$ are as follows: $A_0 = 1$, $A_1 = 2$, $A_2 = 14$, $A_3 = 68$, $A_4 = 406$, $A_5 = 2{,}332$, $A_6 = 13{,}964$, $A_7 = 83{,}848$, etc. The "etc." is puzzling—can any reader discover a closed form expression for A_n?

Also solved by C. Georghiou.

Continue

H-343 *Proposed by Verner E. Hoggatt, Jr. (deceased)*
(Vol. 20, No. 3, August 1982)

Show that every positive integer, m, has a unique representation in the form

$$m = [A_1[A_2[A_3[\ldots[A_n]\ldots],$$

where $A_j = \alpha$ or α^2 for $j = 1, 2, \ldots, n - 1$, and

$$A_n = \alpha^2,$$

where $\alpha = (1 + \sqrt{5})/2$.

Solution by Paul Bruckman, Carmichael, CA

Let $A(k) = [\alpha k]$, $B(k) = [\alpha^2 k]$, $k = 1, 2, 3, \ldots$. Note $A(1) = [\alpha] = 1$ and $B(1) = [\alpha^2] = 2$. Let a "string" denote any composition of functions A or B ending with $B(1)$ [e.g., $A(B(A(B(1))))$]. Let the *length* of a string denote the number n of functions used in the string ($n = 4$ in the example). Let

$$A = \left(A(k)\right)_{k=1}^{\infty},\ B = \left(B(k)\right)_{k=1}^{\infty},\ N = (k)_{k=1}^{\infty}.$$

It is a well-known theorem that $A \cup B = N$, $A \cap B = \emptyset$.

The problem is incorrectly stated, since $1 = A(1)$ is *not* representable by a string. We shall prove that all integers > 1 *are* representable.

We first prove that distinct strings represent distinct positive integers. This is trivially true for $n = 1$, since there is only one number of string-length 1, namely $B(1) = 2$. Also, for $n = 2$, we have

$$A(B(1)) = A(2) = 3 \quad \text{and} \quad B(B(1)) = B(2) = 5.$$

Suppose that all distinct strings of length $\leqslant n$ represent distinct positive integers. Then, if k is the integer represented by any string of length n, we have $A(k) \neq B(k)$, since $A \cap B = \emptyset$. Likewise, $A(k) \neq B(j)$, where j is the integer represented by any string of length less than n. If $A(k) = A(j)$ or $B(k) = B(j)$, then $k = j$, since $A(m)$ and $B(m)$ are one-to-one functions. This is, however, contrary to hypothesis. Thus, all distinct strings of length $\leqslant (n + 1)$ represent distinct integers. It follows by induction that distinct strings represent distinct positive integers.

It remains to show that all positive integers $m > 1$ are thus representable. Suppose that all integers k, with $2 \leqslant k \leqslant m$ are representable. Since $A \cup B = N$, thus, $m + 1 = A(j)$ or $B(j)$ for some integer j with $2 \leqslant j \leqslant m$. Therefore, $m + 1$ is also representable. Since $2 = B(1)$, $3 = A(B(1))$, etc., it follows by induction that all integers $m > 1$ are representable. This completes the proof of the problem (as modified).

Also solved by the proposer and by L. Kuipers, who remarked that the solution is contained in this quarterly, Vol 17 (1979):306-07.

ADVANCED PROBLEMS AND SOLUTIONS

Don't Lose Your Identity

H-344 *Proposed by M. D. Agrawal, Government College, Mandsaur, India (Vol. 20, No. 3, August 1982)*

Prove:

1. $L_k L_{k+3m}^2 - L_{k+4m} L_{k+m}^2 = (-1)^k 5^2 F_m^2 F_{2m} F_{k+2m}$ and
2. $L_k L_{k+3m}^2 - L_{k+2m}^3 = 5(-1)^k F_m^2 (L_{k+4m} + 2(-1)^m L_{k+2m})$.

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

Using the Binet formulas

$$L_n = a^n + b^n \quad \text{and} \quad \sqrt{5}F_n = a^n - b^n$$

and the fact that $ab = -1$,

$$\begin{aligned}
L_k L_{k+3m}^2 - L_{k+4m} L_{k+m}^2 &= (a^k + b^k)(a^{k+3m} + b^{k+3m})^2 - (a^{k+4m} + b^{k+4m})(a^{k+m} + b^{k+m})^2 \\
&= (a^k + b^k)(a^{2k+6m} + 2(-1)^{k+m} + b^{2k+6m}) \\
&\quad - (a^{k+4m} + b^{k+4m})(a^{2k+2m} + 2(-1)^{k+m} + b^{2k+2m}) \\
&= a^{3k+6m} + (-1)^k (a^{k+6m}) + 2(-1)^{k+m}(a^k + b^k) + (-1)^k b^{k+6m} \\
&\quad + b^{3k+6m} - a^{3k+6m} - (-1)^k (a^k b^{2m}) - 2(-1)^{k+m}(a^{k+4m} + b^{k+4m}) \\
&\quad - (-1)^k (a^{2m} b^k) - b^{3k+6m} \\
&= (-1)^k [(a^{k+6m} + b^{k+6m}) + 2(-1)^m (a^k + b^k) \\
&\quad - 2(-1)^m (a^{k+4m} + b^{k+4m}) - (a^k b^{2m} + a^{2m} b^k)].
\end{aligned}$$

Also

$$\begin{aligned}
(-1)^k 5^2 F_m^2 F_{2m} F_{k+2m} &= (-1)^k (a^m - b^m)^2 (a^{2m} - b^{2m})(a^{k+2m} - b^{k+2m}) \\
&= (-1)^k (a^{2m} - 2(-1)^m + b^{2m})(a^{k+4m} - b^k - a^k + b^{k+4m}) \\
&= (-1)^k [a^{k+6m} - a^{2m} b^k - a^{k+2m} + b^{k+2m} - 2(-1)^m a^{k+4m} \\
&\quad + 2(-1)^m b^k + 2(-1)^m a^k - 2(-1)^m b^{k+4m} \\
&\quad + a^{k+2m} - b^{k+2m} - a^k b^{2m} + b^{k+6m}] \\
&= (-1)^k [(a^{k+6m} + b^{k+6m}) + 2(-1)^m (a^k + b^k) \\
&\quad - 2(-1)^m (a^{k+4m} + b^{k+4m}) - (a^k b^{2m} + a^{2m} b^k)].
\end{aligned}$$

This establishes the first formula.

Again using the Binet formulas and the fact that $ab = -1$,

$$\begin{aligned}
L_k L_{k+3m}^2 - L_{k+2m}^3 &= (a^k + b^k)(a^{k+3m} + b^{k+3m})^2 - (a^{k+2m} + b^{k+2m})^3 \\
&= (a^k + b^k)(a^{2k+6m} + 2(-1)^{k+m} + b^{2k+6m}) \\
&\quad - (a^{3k+6m} + 3(-1)^k a^{k+2m} + 3(-1)^k b^{k+2m} + b^{3k+6m}) \\
&= a^{3k+6m} + (-1)^k a^{k+6m} + 2(-1)^{k+m}(a^k + b^k) \\
&\quad + (-1)^k b^{k+6m} + b^{3k+6m} - a^{3k+6m} \\
&\quad - 3(-1)^k(a^{k+2m} + b^{k+2m}) - b^{3k+6m} \\
&= (-1)^k[(a^{k+6m} + b^{k+6m}) + 2(-1)^m(a^k + b^k) - 3(a^{k+2m} + b^{k+2m})].
\end{aligned}$$

Also

$$\begin{aligned}
&5(-1)^k F_m^2(L_{k+4m} + 2(-1)^m L_{k+2m}) \\
&= (-1)^k(a^m - b^m)^2[(a^{k+4m} + b^{k+4m}) + 2(-1)^m(a^{k+2m} + b^{k+2m})] \\
&= (-1)^k(a^{2m} - 2(-1)^m + b^{2m})[(a^{k+4m} + b^{k+4m}) + 2(-1)^m(a^{k+2m} + b^{k+2m})] \\
&= (-1)^k[a^{k+6m} + b^{k+2m} + 2(-1)^m a^{k+4m} + 2(-1)^m b^k - 2(-1)^m a^{k+4m} - 2(-1)^m b^{k+4m} \\
&\quad - 4a^{k+2m} - 4b^{k+2m} + a^{k+2m} + b^{k+6m} + 2(-1)^m a^k + 2(-1)^m b^{k+4m}] \\
&= (-1)^k[(a^{k+6m} + b^{k+6m}) + 2(-1)^m(a^k + b^k) - 3(a^{k+2m} + b^{k+2m})].
\end{aligned}$$

This establishes the second formula.

Also solved by P. Bruckman, W. Janous, L. Kuipers, J. Spraggon, and the proposer.

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The Fibonacci Association and the University of Patras, Greece would like to announce their intentions to jointly sponsor an international conference on Fibonacci numbers and their applications. This conference is tentatively set for late August or early September of 1984. Anyone interested in presenting a paper or attending the conference should contact:

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