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ON THE SOLUTION OF $\left\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\right\}^m G_n = n^k$, BY EXPANSIONS AND OPERATORS

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I. INTRODUCTION

This paper continues the work initiated in the author's joint paper [1] with A. Qadir, in which the authors found the particular solution of the difference equation $(E^2 - E - 1)G_n = n^k$, using two methods, that is, the usual operator method and the method of expansions, eventually establishing an identity involving the Fibonacci numbers F_n defined recursively by $F_1 = F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n, \; n \geqslant 1,$$

the Lucas numbers L_n given by $L_0 = 2$, $L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n, \; n \geqslant 0,$$

and the Sterling numbers of the second kind.

In this paper, the author uses the same two methods to solve a more general difference equation, namely,

$$\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\}^m G_n = n^k,$$

getting an identity involving the Sterling numbers of the second kind, the mth convolved Fibonacci numbers, $F_n^m(p, q)$, where

$$\frac{1}{(1 - px - qx^2)^m} = \sum_{i=0}^{\infty} F_i^m(p, q)x^i$$

and the generalized Lucas numbers, where

$$L_{n+2}(p, q) = pL_{n+1}(p, q) + qL_n(p, q), \; L_0(p, q) = 2, \; L_1(p, q) = p.$$

The plan for this work is as follows. First, in II we find the particular solution of the above-mentioned difference equation by the usual operator method. Then, in III we find the particular solution of the same equation by the method of expansions. Finally, in IV we compare the coefficients of similar powers of n and those of λ, which finally results in the aforesaid identities.

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II. PARTICULAR SOLUTION BY THE METHOD OF OPERATORS

From [1] it is known that

$$\frac{n^k}{E - a} = \sum_{i=0}^{k} \sum_{r=0}^{i} \frac{(-1)^r \binom{k}{i} (r)! S(i, r) n^{k-i}}{(1 - a)^{r+1}}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{i} (r)! S(i, r) n^{k-i}}{(1 - a)^{r+1}}.$$

Where $S(i, r)$ are the Sterling numbers of the second kind, the shift operator E is defined as

$$Ef(n) = f(n + 1)$$

and the difference operator Δ is defined as

$$\Delta f(n) = f(n + 1) - f(n) = (E - 1)f(n).$$

That is, $\Delta = E - 1$.

Therefore,

$$\frac{n^k}{(E - 1 + \lambda a)} = \sum_{i=0}^{k} \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{i} (r)! S(i, r) n^{k-i}}{\lambda^{r+1} a^{r+1}}$$

Also,

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \sum_{s=0}^{k} \sum_{t=0}^{k} \frac{(-1)^{r+t} \binom{k}{i} \binom{k - i}{s} (r)!(t)! S(i, r) S(s, t) n^{k-i-s}}{\lambda^{2+r+t} a^{r+1} b^{t+1}}$$

Letting $\ell = i + s$ implies $\min(\ell) = 0$, $\max(\ell) = k$, so that

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \sum_{\ell=0}^{k} \sum_{t=0}^{k} \frac{(-1)^{r+t} \binom{k}{i} \binom{k - i}{\ell - i} (r)!(t)! S(i, r) S(\ell - i, t) n^{k-\ell}}{\lambda^{2+r+t} a^{1+r} b^{1+t}}$$

Putting $j = r + t$, we have $\min(j) = 0$ and $\max(j) = k$. Now, recall that

$$\binom{k}{i}\binom{k - 1}{\ell - 1} = \binom{k}{\ell}\binom{\ell}{i}$$

and change ℓ to i_2, i to i_1, r to j_1, and j to j_2, to get

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \frac{(-1)^{j_2} \prod_{t=1}^{2} \binom{i_{t+1}}{i_t} (j_t - j_{t-1})! S(i_t - i_{t-1}, j_t - j_{t-1}) n^{k-i_2}}{\lambda^{2+j_2} a^{1+j_1} b^{1+j_2-j_1}}$$

where $i_3 = k$, $i_0 = 0 = j_0$.

Using induction on m, it can be proved that

$$\frac{n^k}{(E - 1 + \lambda a)^m (E - 1 + \lambda b)^m} \tag{2.1}$$

$$= \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m} \binom{i_{t+1}}{i_t} (J_t)! S(I_t, J_t) n^{k-i_{2m}}}{\lambda^{2m+j_{2m}} a^{m-T_{2m-1}} b^{m+T_{2m}}}$$

where $i_{2m+1} = k$, $i_0 = 0 = j_0$, $T_m = \sum_{i=1}^{m} (-1)^i j_i$,

$I_t = i_t - i_{t-1}$ and $J_t = j_t - j_{t-1}$ for every $t > 0$.

Let $G(n, m, k)$ be the particular solution of the difference equation

$$\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\}^m G_n = n^k,$$

and let a, b be the roots of $x^2 = px + q$.

Noting that the left-hand side of (2.1) is symmetric in a, b, we interchange a and b in (2.1) and add the resulting equation to (2.1). Using the fact that $a + b = p$ and $ab = -q$, we get, after a little manipulation,

$$G(n, m, k) \tag{2.2}$$

$$= \frac{1}{2} \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m} \binom{i_{t+1}}{i_t} (J_t)! S(I_t, J_t) L_{T_{2m}+T_{2m-1}} n^{k-i_{2m}}}{\lambda^{2m+j_{2m}} (-q)^{m+T_{2m}}}$$

where $L_s = L_s(p, q)$.

Interchanging a, b in (2.1) and subtracting the resulting equation from (2.1) and dividing both sides by $a - b$, we also have

$$\sum_{i_1=0}^{k} \sum_{i_{2m-1}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m-1}=0}^{k} \frac{\prod_{t=1}^{2m-1} \binom{i_{t+1}}{i_t} (J_t)! S(I_t, J_t) F_{T_{2m}+T_{2m-1}}}{(-q)^{m+T_{2m}}} = 0 \tag{2.3}$$

where $F_s = F_s(p, q)$.

3. PARTICULAR SOLUTION BY THE METHOD OF EXPANSIONS

A Particular Solution of $G(n, m, k)$ is given by

$$G(n, m, k) = \frac{n^k}{(E - 1 + \lambda a)^m (E - 1 + \lambda b)^m} = \frac{n^k}{(\Delta + \lambda a)^m (\Delta + \lambda b)^m}.$$

That is,

$$G(n, m, k) = \frac{n^k}{(\Delta^2 + \lambda p \Delta - \lambda^2 q)^m}, \tag{3.1}$$

where a, b are the roots of $x^2 = px + q$. Since $a + b = p$, $ab = -q$, (3.1) becomes

$$G(n, m, k) = \frac{n^k}{(\Delta^2 + \lambda p \Delta - \lambda^2 q)^m} = \frac{(-1)^m n^k}{\lambda^{2m} q^m \left\{1 - p\left(\frac{\Delta}{q\lambda}\right) - q\left(\frac{\Delta}{q\lambda}\right)^2\right\}^m}$$

$$= \frac{(-1)^m}{\lambda^{2m} q^m} \sum_{i=0}^{\infty} F_i^m(p, q)\left(\frac{\Delta}{q\lambda}\right)^i \cdot n^k$$

where $F_i^m(p, q)$ are the mth convolved Fibonacci numbers.

Therefore,

$$G(n, m, k) = \frac{(-1)^m}{\lambda^{2m} q^m} \sum_{i=0}^{k} \frac{F_i^m(p, q)\Delta^i}{\lambda^i q^i} \sum_{j=0}^{k} S(k, j) \cdot n^{(j)},$$

where $S(k, j)$ are the Sterling numbers of the second kind and

$$n^{(j)} = n(n - 1) \ldots (n - j + 1), \text{ for all } j \geqslant 1,\ n^{(0)} \equiv 1.$$

Therefore,

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{j=0}^{k} \frac{(-1)^m F_i^m(p, q)(j)^{(i)} S(k, j) n^{(j-i)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{j=i}^{k} \frac{(-1)^m (j)^{(i)} F_i^m(p, q) S(k, j) n^{(j-i)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{(-1)^m \binom{j + i}{i} (i)! F_i^m(p, q) S(k, j + i) n^{(j)}}{q^{m+i} \lambda^{2m+i}}.$$

Now, change j to $k - i - j$ in order to reverse the order of summation of j. Then, putting $i + j = \ell$ implies that $\min(\ell) = 0$, $\max(\ell) = k$, so that

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{\ell=0}^{k} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) n^{(k-\ell)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{\ell=0}^{k} \sum_{t=0}^{k-\ell} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) S_t^{k-\ell} n^t}{q^{m+i} \lambda^{2m+i}}$$

where $S_t^{k-\ell}$ are the Sterling numbers of the first kind.

Let us once again reverse the order of summation of t by changing t to $k - \ell - t$. We then let $\ell + t = r$ so that $\min(r) = 0$ and $\max(r) = k$. Then

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{\ell=0}^{k} \sum_{r=0}^{k} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) S_{k-r}^{k-\ell} n^{k-r}}{q^{m+i} \lambda^{2m+i}}.$$

Now, replace ℓ by $k - \ell$ in order to reverse the summation of ℓ. Next, note that

$$S(k, \ell + i) = 0 \text{ if } \ell > k - i \text{ and } S_{k-r}^{\ell} = 0 \text{ if } \ell < k - r.$$

Also, from [2], we have

$$\sum_{\ell=k-r}^{\ell=k-i} \binom{\ell + i}{i} S(k, \ell + i) S_{k-r}^{\ell} = \binom{k}{r} S_{(r, i)}$$

Hence, writing i_{2m} for r and j_{2m} for i, we obtain

$$G(n, m, k) = \sum_{i_{2m}=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^m F_{j_{2m}}^m (p, q) \binom{k}{i_{2m}} (j_{2m})! S(i_{2m}, j_{2m}) n^{k-i_{2m}}}{q^{m+j_{2m}} \lambda^{2m+j_{2m}}}. \tag{3.2}$$

4. THE DERIVATION OF THE IDENTITY

Equating the coefficients of similar powers of n from (2.2) and (3.2), and dividing both sides of the resulting equation by the common factor

$$\binom{k}{i_{2m}}$$

we have

$$\frac{1}{2} \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m-i} \binom{i_{t+1}}{i_t} (J_t)! S(I_t, J_t) L_{T_{2m} + T_{2m-1}}}{\lambda^{2m+j_{2m}} (-q)^{m+T_{2m}}} \tag{4.1}$$

$$= \sum_{j_{2m}=0}^{k} \frac{(-1)^m (j_{2m})! \binom{k}{i_{2m}} S(i_{2m}, j_{2m}) F_{j_{2m}}^m}{q^{m+j_{2m}} \lambda^{2m+j_{2m}}}$$

where $F_{j_{2m}}^m \equiv F_{j_{2m}}^m (p, q)$.

Finally, equating the coefficients of similar powers of λ in (4.1), we obtain

$$\sum_{i_1=0}^{k} \sum_{i_{2m-1}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m-1}=0}^{k} \frac{\prod_{t=1}^{2m-1} \binom{i_{t+1}}{i_t} (J_t)!(-1)^{T_{2m-1}} S(I_t, J_t) L_{T_{2m}+T_{2m-1}}}{q^{T_{2m-1}}} \tag{4.2}$$

$$= 2(j_{2m})! S(i_{2m}, j_{2m}) F^m_{j_{2m}}.$$

Equating (2.3) and (4.2) gives the identities we wanted to derive.

REFERENCES

1. H. N. Malik & A. Qadir. "Solution of Pseudo-Periodic Difference Equations." In *A Collection of Manuscripts Related to the Fibonacci Sequence: 18th Anniversary Volume*, ed. by V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. Santa Clara, Calif.: The Fibonacci Association, 1980, pp. 179-186.
2. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968, p. 204.

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