◆◇◆◇◆

ON FIBONACCI NUMBERS OF THE FORM PX^2, WHERE P IS PRIME

NEVILLE ROBBINS

San Francisco State University, San Francisco, CA 94132

(Submitted April 1982)

INTRODUCTION

Let p denote a prime, n a natural number, $F(n)$ the nth Fibonacci number. Consider the equation:

$$F(n) = px^2. \tag{*}$$

In [3], J. H. E. Cohn proved that for $p = 2$, the only solutions of (*) are

(i) $n = 3$, $x^2 = 1$

and

(ii) $n = 6$, $x^2 = 4$.

In [8], R. Steiner proved that for $p = 3$, the only solution of (*) is $n = 4$, $x^2 = 1$. Call a solution of (*) trivial if $x = 1$. In this article, we solve (*) for all odd p such that $p \equiv 3 \pmod 4$ or $p < 10{,}000$. Except for $p = 3{,}001$, all solutions obtained are trivial. $L(n)$ denotes the nth Lucas number.

Definition 1

$z(p)$ is the Fibonacci entry point of p, that is,

$$z(p) = \min\{k : k > 0 \text{ and } p \mid F(k)\}.$$

Definition 2

$y(p)$ is the least prime factor of $z(p)$.

PRELIMINARY RESULTS

(1) $F(2m) = F(m)L(m)$

(2) $(F(m), L(m)) \mid 2$

(3) If $\prod_{i=1}^{m} a_i = b$ and the a_i are pairwise coprime, then each $a_i = b_i^n$, where the b_i are pairwise coprime and $\prod_{i=1}^{m} b_i = b$.

(4) $p \mid F(n)$ iff $z(p) \mid n$

(5) $F(m) = x^2$ implies $m = 1$, 2, or 12

(6) $F(m) = 2x^2$ implies $m = 3$ or 6

(7) $L(m) = x^2$ implies $m = 1$ or 3

(8) $L(m) = 2x^2$ implies $m = 6$

(9) $p \geqslant 5$ implies $y(p) \leqslant p$

(10) $(F(m), F(km)/F(m)) | k$

(11) $p \equiv 3 \pmod 4$ implies $z(p)$ is even

(12) $p | F(p)$ iff $p = 5$

Remarks: (5) through (8) are Theorems 1 through 4 in [3]. (9) follows from Theorem 3 in [7]. (10) is Lemma 16, p. 224 in [6]. The other preliminary results are elementary or well known.

THE MAIN THEOREMS

Theorem 1

If $n = 2m$, then the unique solution of (*) is $p = 3$, $n = 4$, $x^2 = 1$.

Proof: Hypothesis and (1) imply $F(m)L(m) = px^2$. Now (2) and (3) imply $F(m)$ or $L(m)$ is a square or twice a square. By (5), (6), (7), and (8), we have $m = 1$, 2, 3, 6, or 12. The only case which yields a solution of (*) is $m = 2$, so that $n = 4$, $p = 3$, $x^2 = 1$.

Corollary 1

If $p \equiv 3 \pmod 4$, then the unique solution of (*) is $p = 3$, $n = 4$, $x^2 = 1$.

Proof: Follows from hypothesis, (11), (4), and Theorem 1.

Theorem 2

If n is odd, then any solution of (*) requires that $n = z(p) = q$, a prime, unless $n = x^2 = 25$ and $p = 3{,}001$.

Proof: Hypothesis and (4) imply that $z(p)$ is odd. By [5, pp. 643-45], we have $n = z(p) \equiv \pm 1 \pmod 6$, so that $n = q^k m$, where $q \geqslant 5$, $k \geqslant 1$, and each prime factor of m exceeds q. If $q | F(m)$, then (4) and Definition 2 imply $y(q) | m$. But (9) implies $y(q) \leqslant q$, a contradiction. Therefore,

$$(q, F(m)) = 1.$$

Now (*) implies $px^2 = F(m) * F(q^k m)/F(m)$. Let $d = (F(m), F(q^k m)/F(m))$. (10) implies $d | q^k$. Therefore, the only possible prime divisor of d is q. But, since $(q, F(m)) = 1$, we have $d = 1$. Since $m < n$, (4) implies $(p, F(m)) = 1$, so $F(m)$ is a square. Since m is odd, (5) implies $m = 1$, so that $n = q^k$.

Therefore, $$px^2 = F(q^{k-1}) * F(q^k)/F(q^{k-1}).$$

Let $d' = (F(q^{k-1}), F(q^k)/F(q^{k-1}))$. (10) implies $d' = 1$ or q. If $d' = 1$, then, since $n = z(p) = q^k$, we have $(p, F(q^{k-1})) = 1$. Once again, this implies that $F(q^{k-1}) = 1$, hence $q^{k-1} = 1$ and $k = 1$. If $d' = q$, then (12) implies $q = 5$, so that (3) implies $F(5^{k-1}) = 5x_1^2$. We have

$$x_1^2 = F(5^{k-1})/F(5) = P_{5^{k-2}}(11)$$

in the notation of [5]. By Theorem 3 in [4], this implies $5^{k-2} = 1$, i.e., $n = 25$, and thus $p = 3{,}001$.

Theorem 3

If $2 < p < 10{,}000$, then the unique nontrivial solution of (*) is $p = 3{,}001$, $n = x^2 = 25$; all other solutions are trivial with

$$(n, p) = (4, 3), (5, 5), (11, 89), (13, 233), \text{ or } (17, 1{,}571).$$

Proof: If n is even, then Theorem 1 implies $(n, p) = (4, 3)$. If n is odd and $p \neq 3{,}001$, then Theorem 2 implies $n = z(p) = q$, a prime. We therefore consider all p such that $5 \leqslant p < 10{,}000$, $p \neq 3{,}001$, and $q = z(p)$ is an odd prime. If $q \leqslant 229$, namely for p = 5, 13, 37, 73, 89, 113, 149, 157, 193, 233, 269, 277, 313, 353, 389, 397, 457, 557, 677, 953, 1,069, 1,597, 2,221, 2,417, 2,749, 2,789, 4,013, 4,513, 5,737, 6,673, or 8,689, the conclusion follows from the examination of the prime factorization of $F(q)$ in [2]. According to [1], there are 101 primes, p, such that $p < 10{,}000$ and $q = z(p)$ is a prime exceeding 229. For each such p, to show that $F(q) \neq px^2$, it suffices to find an odd prime modulus, t, such that $F(q)/p$ is a quadratic nonresidue (mod t). The results are listed in Table 1. For each p, the corresponding t is the least required prime modulus. In each case, $t \leqslant 19$.

TABLE 1

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
613	307	3	2	1	1	2
673	337	19	5	8	12	3
733	367	7	1	5	3	3
757	379	3	2	1	1	2
877	439	5	1	2	3	3
997	499	3	2	1	1	2
1093	547	3	2	1	1	2
1153	577	7	1	5	3	3
1213	607	11	2	3	4	8
1237	619	3	2	1	1	2
1453	727	7	6	4	2	5
1657	829	3	2	1	1	2
1753	877	3	2	1	1	2
1873	937	7	6	4	2	5
1877	313	3	1	2	2	2

TABLE 1 (continued)

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
1933	967	7	6	1	1	6
1949	487	3	1	2	2	2
1993	997	3	2	1	1	2
2017	1009	5	4	2	3	2
2137	1069	3	2	1	1	2
2237	373	7	5	4	2	3
2309	577	3	1	2	2	2
2333	389	5	4	3	2	3
2437	1237	3	2	1	1	2
2557	1279	5	1	2	3	3
2593	1297	7	1	3	5	5
2777	463	3	1	2	2	2
2797	1399	5	1	2	3	3
2857	1429	3	2	1	1	2
2909	727	3	1	2	2	2
2917	1459	3	2	1	1	2
3217	1609	5	4	2	3	2
3253	1627	3	2	1	1	2
3313	1657	7	6	2	4	3
3517	1759	5	1	2	3	3
3557	593	3	1	2	2	2
3733	1867	3	2	1	1	2
4057	2029	3	2	1	1	2
4177	2089	5	4	2	3	2
4273	2137	11	2	5	9	7
4349	1087	3	1	2	2	2
4357	2179	3	2	1	1	2
4637	773	13	11	9	3	7
4733	263	3	1	2	2	2
4909	409	7	6	2	4	3
4933	2467	3	2	1	1	2
5009	313	3	1	2	2	2
5077	2539	3	2	1	1	2
5113	2557	3	2	1	1	2
5189	1297	3	1	2	2	2
5233	2617	7	6	4	2	5
5297	883	7	2	5	3	6
5309	1327	3	1	2	2	2
5381	269	7	2	5	3	6
5413	2707	3	2	1	1	2
5437	2719	5	1	2	3	3
5653	257	17	5	9	2	10
5897	983	3	1	2	2	2
6037	3019	3	2	1	1	2
6073	3037	3	2	1	1	2
6133	3067	3	2	1	1	2
6217	3109	3	2	1	1	2
6269	1567	3	1	2	2	2
6337	3169	5	4	2	3	2

TABLE 1 (continued)

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
6373	3187	3	2	1	1	2
6397	457	13	8	1	1	8
6637	3319	5	1	2	3	3
6737	1123	7	2	3	5	3
6917	1153	3	1	2	2	2
6997	3499	3	2	1	1	2
7057	3529	5	4	2	3	2
7109	1777	3	1	2	2	2
7213	3607	17	13	5	7	6
7393	3697	11	2	1	1	2
7417	3709	3	2	1	1	2
7477	3739	3	2	1	1	2
7537	3769	5	4	2	3	2
7753	3877	3	2	1	1	2
7817	1303	3	1	2	2	2
7933	3967	13	8	3	9	7
8053	4027	3	2	1	1	2
8317	4159	5	1	2	3	3
8353	4177	11	2	4	3	6
8369	523	5	2	4	4	3
8573	1429	5	4	3	2	3
8677	4339	3	2	1	1	2
8713	4357	3	2	1	1	2
8753	1459	5	1	3	2	2
8861	443	5	2	1	1	2
8893	4447	7	1	3	5	5
9013	4507	3	2	1	1	2
9133	4567	11	2	3	4	8
9277	4639	5	1	2	3	3
9377	521	3	1	2	2	2
9397	4699	3	2	1	1	2
9497	1583	3	1	2	2	2
9677	1613	7	2	3	5	3
9697	373	3	2	1	1	2
9817	4909	3	2	1	1	2
9949	829	3	2	1	1	2
9973	4987	3	2	1	1	2

CONCLUDING REMARKS

According to [2], additional trivial solutions exist (corresponding to larger p) for n = 23, 29, 43, 83, 131, 137, 359, 431, 433, 449, 509, and 569. It remains to be decided whether (i) any nontrivial solutions exist apart from those already known, and/or (ii) infinitely many p exist having trivial solutions.

REFERENCES

1. Brother U. Alfred. *Tables of Fibonacci Entry Points.* Santa Clara, Calif.: The Fibonacci Association, 1965.
2. John Brillhart. "Factorizations of u_n, n odd, $3 \leqslant n \leqslant 997$." (Private communication.)
3. J. H. E. Cohn. "Square Fibonacci Numbers, Etc." *The Fibonacci Quarterly* 2, No. 2 (1964):109-13.
4. J. H. E. Cohn. "Eight Diopnantine Equations." *Proc. London Math. Soc.* (3) 16 (1966):153-66.
5. J. H. E. Cohn. "Squares in Some Recurrent Sequences." *Pac. J. Math.* 41 (1972):631-46.
6. John H. Halton. "On the Divisibility Properties of Fibonacci Numbers." *The Fibonacci Quarterly* 4, No. 3 (1966):217-40.
7. S. Mohanty. "The Number of Primes Is Infinite." *The Fibonacci Quarterly* 16, No. 5 (1978):381-84.
8. R. Steiner. "On Fibonacci Numbers of the Form $x^2 + 1$." In *A Collection of Manuscripts Related to the Fibonacci Sequence*, ed. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. Santa Clara, Calif.: The Fibonacci Association, 1980, pp. 208-10.

◆◇◆◇◆