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WAITING TIMES AND GENERALIZED FIBONACCI SEQUENCES

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(Submitted November 1981)

1. INTRODUCTION AND SUMMARY

Suppose we consider the following experiment: Toss a coin until we observe two heads in succession for the first time. One may ask for the probability of this event. Intuitively, one feels that the solution to this problem may be related to the Fibonacci sequence; and, in fact, this is so. More generally, one may be interested in finding the probability distribution of the waiting time to find r heads in succession for the first time. As one may guess, these results contain generalized Fibonacci, Tribonacci, ..., sequences. This problem was studied by Turner [8], who expressed the probability distribution in terms of generalized Fibonacci-T sequences which, in turn, were expressed in terms of generalized Pascal-T triangles. In this paper, we will express the probability distribution of this waiting time as a difference of two sums (Proposition 2.1). This result enables us to express Fibonacci numbers, Tribonacci numbers, etc., and their generalizations as sums of weighted binomial coefficients.

In probability literature (Feller [2]), the probability generating functions of waiting times of this type are well known. We derive Proposition 2.1 from one of these generating functions. In Section 3 we illustrate how one can obtain further generalizations of Fibonacci-T sequences by using the probability generating functions of the waiting times associated with different events of interest. Finally, starting with the generating function, we obtain new formulas for Tribonacci numbers.

2. THE PROBABILITY DISTRIBUTIONS OF WAITING TIMES

Suppose there are k possible outcomes on each trial (denoted by E_1, E_2, ..., E_k) with probabilities π_1, π_2, ..., π , respectively, such that $\pi_i \geqslant 0$ and $\pi_1 + \pi_2 + \cdots + \pi_k = 1$. At each trial, exactly one of the outcomes is observed. After n independent trials, we are interested in finding the

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Research sponsored jointly by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Department of Energy under contract W-7405-eng-26 with Union Carbide Corporation, the Office of Nuclear Regulatory Research of the Nuclear Regulatory Commission, and the Tennessee Technological University in Cookeville, Tennessee.

probability of the first occurrence of r specified outcomes in succession. Let $\underline{E}_r$ denote this event, and W_r denote the waiting time for the first occurrence of $\underline{E}_r$. We are interested in the distributional properties of W_r.

Suppose $\underline{E}_r = \{E_1 E_1 \ldots E_1\}$, which corresponds to the occurrence of the same outcome E_1, r times in a row. Then we have the following:

Proposition 2.1

The probability distribution of the discrete random variable W_r, denoted by f_{n+r}, is given by

$$\begin{aligned} P[W_r = n + r] &= \pi_1^r \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{j} ((1 - \pi_1)\pi_1^r)^j \\ &\quad - \pi_1^{r+1} \sum_{j=0}^{\infty} (-1)^j \binom{n - 1 - jr}{j} ((1 - \pi_1)\pi_1^r)^j, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where we define $\binom{m}{k} = 0$ if $m < k$ or $m < 0$.

The derivation of this proposition will be given in a later section. We discuss the generalities of this result now. If there are two possible outcomes (i.e., $k = 2$) with $\pi_1 = \pi_2 = \frac{1}{2}$, then we define

$$\beta_{n,r} = \begin{cases} 1 & n = 0 \\ 2^{n+r} P[W_r = n + r] = A_{n,r} - A_{n-1,r}, & n \geqslant 1 \end{cases} \tag{2.2}$$

where

$$A_{n,r} = 2^n \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{r} (1/2^{(r+1)j}),$$

with

$$A_{j,r} = 2^j, \text{ for } 0 \leqslant j \leqslant r. \tag{2.3}$$

We shall show later that the sequences $\{\beta_{n,r}\}$ are generalized Fibonacci sequences. Specifically, for $r = 2$, $\{\beta_{n,2}\}$ is the Fibonacci sequence given by 1, 1, 2, 3, 5, 8, 13, ... For $r = 3$, we have the so-called Tribonacci sequence (Feinberg [1]), given by 1, 1, 2, 4, 7, 13, 24, 44, ... For $r = 4$, one can verify that

$$\beta_{n+4,4} = \beta_{n+3,4} + \beta_{n+2,4} + \beta_{n+1,4} + \beta_{n,4}, \tag{2.4}$$

and the sequence $\{\beta_{n,4}\}$ is given by 1, 1, 2, 4, 8, 15, ... For general r, we have

$$\beta_{n+r,r} = \beta_{n+r-1,r} + \beta_{n+r-2,r} + \cdots + \beta_{n,r}, \tag{2.5}$$

which is an rth order Fibonacci-T sequence.

If we leave k unspecified but still require $\pi_1 = \pi_2 = \cdots = \pi_k = 1/k$, then we can define

$$\beta_{n,r}^{(k)} = k^{n+r} [W_r = n + r] \tag{2.6}$$

so that, using Proposition 2.1, we get

$$\beta_{n,r}^{(k)} = A_{n,r}^{(k)} - A_{n+1,r}^{(k)} \tag{2.7}$$

where
$$A_{n,r}^{(k)} = k^n \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{j} \left[\frac{(k-1)}{(k^{r+1})}\right]^j. \tag{2.8}$$

We prove in Section 3 that

$$\beta_{n+r,r}^{(k)} = (k-1)\left[\beta_{n+r-1,r}^{(k)} + \beta_{n+r-2,r}^{(k)} + \cdots + \beta_{n,r}^{(k)}\right] \tag{2.9}$$

with the boundary conditions

$$\beta_{r,r}^{(k)} = 1 \quad \text{and} \quad \beta_{s,r}^{(k)} = 0 \quad \text{for } s < r;$$

and for the special case $k = 2$, (2.9) gives the recursion satisfied by the rth order Fibonacci-T sequence given in (2.5). For $r = 2$ and $k = 3$, the sequence $\{\beta_{n,2}^{(3)}\}$ is given by 1, 2, 6, 16, 44, 120, ... For $r = 3$ and $k = 3$, the sequence $\{\beta_{n,3}^{(3)}\}$ is given by 1, 2, 6, 18, 52, 152, 444, ...

3. THE PROBABILITY GENERATING FUNCTIONS OF WAITING TIMES

In this section we shall give a derivation of Proposition 2.1, starting from the probability generating function of the waiting times for recurrent events and then prove equation (2.9). Following Feller [2], the generating function given for binomial processes can easily be extended to multinomial processes, for the events of type $\underline{E}_r$ considered in this paper. In particular, the probability generating function of the first occurrence of $\underline{E}_r$ discussed in Section 2, is given by

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} s^{n+r} P[W_r = n + r] = \frac{\pi_1^r s^r (1 - \pi_1 s)}{1 - s + (1 - \pi_1)\pi_1^r s^{r+1}} \\ &= \frac{\pi_1^r s^r}{1 - s + (1 - \pi_1^r)\pi_1 s^{r+1}} - \frac{\pi_1^{r+1} s^{r+1}}{1 - s + (1 - \pi_1^r)\pi_1 s^{r+1}}, \\ &= \text{(i)} - \text{(ii)}. \end{aligned} \tag{3.1}$$

Let $\theta = (1 - \pi_1)\pi_1^r$, then

$$\begin{aligned} \text{(i)} &= \frac{\pi_1^r s^r}{1 - s(1 - s^r\theta)} = \pi_1^r s^r [1 + s(1 - s^r\theta) + s^2(1 - s^r\theta)^2 + \cdots \\ &\quad + s^r(1 - s^r\theta)^r + \cdots + s^{(j-1)r}(1 - s^r\theta)^{(j-1)r} + \cdots]. \end{aligned} \tag{3.2}$$

In (3.2), s^{jr} appears only in the following $(j - 1)$ terms:

$$\pi_1^r s^{jr}(1 - s^r\theta)^{(j-1)r},\ \pi_1^r s^{(j-1)r}(1 - s^r\theta)^{(j-2)r},\ \ldots\ \pi_1^r s^{2r}(1 - s^r\theta)^r;$$

and the coefficient of s^{jr} in (i) is given by

$$\left\{\binom{(j-1)r}{0} - \binom{(j-2)r}{1}\theta + \binom{(j-3)r}{2}\theta^2 \cdots + (-1)^{j-2}\binom{r}{j-2}\theta^{j-2}\right\}\pi_1^r. \tag{3.3}$$

More generally, $s^{jr+\ell}$, $0 \leqslant \ell \leqslant r-1$, appears in (3.2) only in the following $(j-1)$ terms:

$$\pi_1^r s^{jr+\ell}(1 - s^r\theta)^{(j-1)r+\ell},\ \pi_1^r s^{(j-1)r+\ell}(1 - s^r\theta)^{(j-2)r+\ell},$$

$$\pi_1^r s^{(j-2)r+\ell}(1 - s^r\theta)^{(j-3)r+\ell},\ \ldots,\ \pi_1^r s^{2r+\ell}(1 - s^r\theta)^{r+\ell};$$

and the coefficient of $s^{jr+\ell}$, $0 \leqslant \ell \leqslant r-1$, in (3.2) is given by

$$\left\{\binom{(j-1)r+\ell}{0} - \binom{(j-2)r+\ell}{1}\theta + \binom{(j-3)r+\ell}{2}\theta^2 \cdots + (-1)^{j-2}\binom{r+\ell}{j-2}\theta^{j-2}\right\}\pi_1^r. \tag{3.4}$$

Since f_{n+r} is equal to the sum of the coefficients of s^{n+r} in (i) and (ii), taking $n = (j-1)r+\ell$ in the above, we obtain:

$$f_{n+r} = \left\{\binom{n}{0} - \binom{n-r}{1}\theta + \binom{n-2r}{2}\theta^2 \cdots\right\}\pi_1^r - \left\{\binom{n-1}{0} - \binom{n-1-r}{1}\theta + \binom{n-1-2r}{2}\theta^2 \cdots\right\}\pi_1^{r+1}, \tag{3.5}$$

which proves Proposition 2.1.

The probability generating function given by (3.1) can also be written in the form

$$F(s) = 1\bigg/\left[1 + (1-s)\left[\frac{1}{s\pi_1} + \left(\frac{1}{s\pi_1}\right)^2 + \cdots + \left(\frac{1}{s\pi_1}\right)^r\right]\right], \tag{3.6}$$

which may be recognized as a special case of the probability generating function discussed by Johnson [5] and Johnson & Kotz [6]. In order to summarize these results, we need to introduce some notation.

Returning to the situation introduced in Section 2, suppose we are interested in a specific event $\underline{E}_r$ of length r (or r independent outcomes). We shall now obtain the probability generating function for the waiting time, W_r, which denotes the first occurrence associated with the event $\underline{E}_r$. As a first step, we introduct the definition of the critical points of $\underline{E}_r$, as defined by Johnson [5].

Definition: A critical point of $\underline{E}_r$ is defined as the position between two labels, such that the subsequence of labels up to that position is identical to the subsequence of labels of the same length concluding the pattern. Also, a critical point always follows the last trial at which event $\underline{E}_r$ occurs.

As an illustration, suppose we toss a coin so that we have the two possible outcomes, Heads and Tails, denoted by the labels H and T, respectively. For a given pattern like HTHTH, we can observe three critical points. Since the last trial completes the pattern, it precedes a critical point. At the third trial of this pattern, we have a H, and the subsequence HTH up to the third trial is the same as the subsequence at the end of the pattern and, hence, the third trial precedes a critical point. And finally, at the first trial of this pattern we have a H, and we have a H at the end, so the first trial also precedes a critical point.

Let us consider another pattern E_7, defined by HHTHHHT, which has only two critical points. For this pattern, the seventh trial (by definition) and the third trial precede the two critical points.

More generally, let the event of interest, $\underline{E}_r$, have $c(1 \leqslant c \leqslant r)$ critical points. Let $a_{\alpha t}$ denote the number of outcomes E_α observed up to the tth critical point, for $\alpha = 1, 2, \ldots, k$ and $t = 1, 2, \ldots, c$. Then the probability generating function $F(s)$ of W_r, as given by Johnson [5], is

$$F(s) = 1 \Big/ \left[1 + (1 - s)\sum_{t=1}^{c} \frac{1}{\left(s^{a_{1t} + a_{2t} + \cdots + a_{kt}}\right)} \left\{\prod_{\alpha=1}^{k} \pi_\alpha^{-a_{\alpha t}}\right\}\right]. \tag{3.7}$$

Special Cases

(1) When the event of interest $\underline{E}_r$ is given by a succession of r identical events E_1, then there are r critical points associated with this event; and associated with the first critical point, we have

$$a_{11} = 1,\ a_{21} = 0,\ \ldots,\ a_{k1} = 0,$$

and associated with the tth critical point, we have

$$a_{1t} = t,\ a_{\alpha t} = 0,\ \alpha = 2,\ \ldots,\ k \text{ for } t = 2,\ \ldots,\ r.$$

In this case, the probability generating function of the event of length r, given by $E_1 E_1 \ldots E_1$ reduces to

$$F(s) = 1 \Big/ \left[1 + (1 - s)\sum_{t=1}^{r} \frac{1}{s^t} \frac{1}{\pi_1^t}\right], \tag{3.8}$$

which agrees with (3.6).

Next, taking $\pi_1 = 1/k$, we shall derive (2.9). We have

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} s^{n+r} P[W_r = n + r] \\ &= \sum_{n=0}^{\infty} (s/k)^{n+r} \beta_{n,r}^{(k)} && \text{from (2.6)} \\ &= 1 \Big/ \left[1 + (1 - s)\left(\frac{k}{s} + \frac{k^2}{s^2} + \cdots + \frac{k^r}{s^r}\right)\right] && \text{from (3.8)} \\ &= 1 \Big/ \left[\frac{k^r}{s^r} - (k - 1)\left(1 + \frac{k}{s} + \frac{k^2}{s^2} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\right]. \end{aligned}$$

Therefore, we have the relation

$$\left[\sum_{n=0}^{\infty} (s/k)^{n+r} \beta_{n,r}^{(k)}\right]\left[\frac{k^r}{s^r} - (k - 1)\left(1 + \frac{k}{s} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\right] = 1,$$

From which it follows that

$$\sum_{n=0}^{\infty}(s/k)^n\beta_{n,r} = 1 + (k-1)\left(1 + \frac{k}{s} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\sum_{n=0}^{\infty}(s/k)^{n+r}\beta_{n,r}^{(k)}.$$

Equating the coefficients of s^{n+r}, on both sides, we find

$$\beta_{n+r,r}^{(k)} = (k-1)\left[\beta_{n+r-1,r}^{(k)} + \beta_{n+r-2,r}^{(k)} + \cdots + \beta_{n,r}^{(k)}\right],$$

which proves (2.9).

(2) Let the event of interest be $E_1E_2\ldots E_k$, which is of length k, and the outcomes occur in the specified order. This event has only one critical point, and

$$a_{11} = 1 = a_{21} = \cdots = a_{k1},$$

and all others are zero. In this special case, the probability generating function is given by

$$F(s) = 1\bigg/\left[1 + (1-s)\left(\frac{1}{s^k\pi_1\ldots\pi_k}\right)\right]. \tag{3.9}$$

(3) Let $k = 2$ and the event of interest be $E_1E_1E_1$ (of length 3) and let $\pi_1 = \frac{1}{2} = \pi_2$. In this case, there are $c = 3$ critical points and

$$a_{11} = 1,\ a_{12} = 2,\ a_{13} = 3,$$

$$a_{21} = 0,\ a_{22} = 0,\ a_{23} = 0.$$

With these values,

$$\begin{aligned}\sum_{n=0}^{\infty}s^{n+3}P[W_3 = n+3] = F(s) &= 1\bigg/\left[1 + (1-s)\sum_{t=1}^{3}(2/s)^t\right]\\ &= 1\bigg/\left[1 + (1-s)\left(\frac{2}{s} + \frac{4}{s^2} + \frac{8}{s^3}\right)\right] \qquad (3.10)\\ &= \frac{s^3}{s^3 + 2(1-s)(s^2 + 2s + 4)}.\end{aligned}$$

From this, we obtain

$$\sum_{n=0}^{\infty}t^{n+3}2^{n+3}P[W_3 = n+3] = F(2t) = t^3/[1 - t - t^2 - t^3],$$

and, as defined in (2.2),

$$2^{n+3}P[W_3 = n+3] = \beta_{n,3}, \tag{3.11}$$

which are the Tribonacci numbers.

From this generating function of the Tribonacci numbers, we obtain a representation for $\beta_{n,3}$ in terms of trigonometric functions, which is stated in the following proposition.

Proposition 3.1

The Tribonacci numbers $\beta_{n,3}$ are given by

$$\beta_{n,3} = \frac{1}{(c-1)(c+3)}\left[c^{1+(n/2)}\left\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{c^{3/2}\sin n\theta}{\sin\theta}\right\} - \frac{1}{c^{n-1}}\right]$$

for $n = 2, 3, \ldots$, where

$$c = (1/3)[(\sqrt{297}+17)^{1/3} - (\sqrt{297}-17)^{1/3} - 1]$$

and $\theta = \pi - \text{Arc}\sin(\sqrt{3-c^2})/2$, and $\beta_{0,3}$ and $\beta_{1,3}$ are defined to be equal to 1. From (3.11), we note that $\beta_{n,3}$ is given by the coefficient of t^{n-1} in $1/(1-t-t^2-t^3)$. In order to find this coefficient, we use partial fractions given by

$$\frac{1}{1-t-t^2-t^3} = \frac{C}{(c-t)} + \frac{D}{(d-t)} + \frac{G}{(g-t)}.$$

Let c, d, and g denote the real and the complex conjugate roots of the cubic $1 - t - t^2 - t^3 = 0$, given by

$$c = (1/3)(\gamma - \delta - 1),$$

$$d = (-1/6)(\gamma - \delta - 2) + (\sqrt{3}/6)i(\gamma + \delta) = (1/\sqrt{c})e^{i\theta},$$

and $$g = (1/\sqrt{c})e^{-i\theta}$$

where $\gamma = (\sqrt{297}+17)^{1/3}$, $\delta = (\sqrt{297}-17)^{1/3}$, and $i = \sqrt{-1}$. Now, C, D, and G can be expressed in terms of c, d, and g, and we obtain

$$\begin{aligned}\frac{1}{1-t-t^2-t^3} &= \frac{1}{(d-c)(g-c)c}\left[1 + \frac{t}{c} + \frac{t^2}{c^2} + \frac{t^3}{c^3} + \cdots + \frac{t^{n-1}}{c^{n-1}} + \cdots\right]\\ &+ \frac{1}{(c-d)(d-g)d}\left[1 + \frac{t}{d} + \frac{t^2}{d^2} + \cdots + \frac{t^{n-1}}{d^{n-1}} + \cdots\right]\\ &+ \frac{1}{(c-g)(d-g)g}\left[1 + \frac{t}{g} + \frac{t^2}{g^2} + \cdots + \frac{t^{n-1}}{g^{n-1}} + \cdots\right].\end{aligned}$$

Therefore, $\beta_{n,3}$ can be obtained as the coefficient of t^{n-1}, given by

$$\begin{aligned}\beta_{n,3} &= \frac{1}{c(c-d)(g-c)}\left[-\frac{1}{c^{n-1}} + \frac{c(g-c)}{(g-d)d^n} - \frac{c(c-d)}{(d-g)g^n}\right]\\ &= \frac{-1}{c(c-d)(g-c)}\left[\frac{1}{c^{n-1}} + \frac{(cg-c^2)}{(d-g)}c^ng^n + \frac{(c^2-cd)}{(d-g)}c^nd^n\right]\\ &\qquad\qquad \text{(here we use the fact that } cdg = 1)\\ &= \frac{-1}{c(c-d)(g-c)}\,\frac{1}{c^{n-1}}\\ &\quad + \frac{1}{c(c-d)(g-c)}\left[\frac{c^{n+1}(g^{n+1}-d^{n+1})}{(g-d)} - \frac{c^{n+2}(g^n-d^n)}{(g-d)}\right].\end{aligned}$$

The following identities between the roots c, d, and g can be verified.

$$\text{(i)} \quad c(c - d)(g - c) = (c - 1)(c + 3)$$

and

$$\text{(ii)} \quad \frac{\sin(k + 1)\theta}{\sin \theta} = \frac{d^{k+1} - g^{k+1}}{d - g},$$

where θ is as defined in Proposition (3.1).

Using these properties, we find

$$(c - 1)(c + 3)\beta_{n,3} = c^{1+(n/2)}\left\{\frac{\sin(n + 1)\theta}{\sin \theta} - \frac{c^{3/2} \sin n\theta}{\sin \theta}\right\} - \frac{1}{c^{n-1}},$$

for $n = 2, 3, \ldots$. This representation corresponds to the "Golden Number" representation of the Fibonacci numbers.

4. REMARKS

We wish to thank a referee for bringing to our attention the article by Philippou & Muwafi [6], which also deals with the waiting time problem for the kth consecutive success of a Bernoulli process. There is not much of an overlap with our results, and the references cited by these authors may be of historical interest to the reader.

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