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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

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Send all communications regarding ELEMENTARY PROBLEMS and SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-514 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2$ for $n = 5, 6, 7, \ldots$.

B-515 *Proposed by Walter Blumberg, Coral Springs, FL*

Let $Q_0 = 3$, and for $n \geqslant 0$, $Q_{n+1} = 2Q_n^2 + 2Q_n - 1$. Prove that $2Q_n + 1$ is a Lucas number.

B-516 *Proposed by Walter Blumberg, Coral Springs, FL*

Let U and V be positive integers such that $U^2 - 5V^2 = 1$. Prove that UV is divisible by 36.

B-517 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

Find all n such that $n! + (n + 1)! + (n + 2)!$ is the square of an integer.

B-518 *Proposed by Herta T. Freitag, Roanoke, VA*

Let the measures of the legs of a right triangle be $F_{n-1}F_{n+2}$ and $2F_nF_{n+1}$. What feature of the triangle has $F_{n-1}F_n$ as its measure?

B-519 *Proposed by Herta T. Freitag, Roanoke, VA*

Do as in Problem B-518 with each Fibonacci number replaced by the corresponding Lucas number.

SOLUTIONS

Lucas Addition Formula

B-490 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that the arithmetic mean of $L_{2n}L_{2n+3}$ and $5F_{2n}F_{2n+3}$ is always a Lucas number.

Solution by J. Suck, Essen, GERMANY

This is an instance of the addition formula

$$2L_{m+n} = L_mL_n + 5F_mF_n,\ m, n \in \mathbf{Z}, \tag{*}$$

a companion to $2F_{m+n} = F_mL_n + L_mF_n$ (compare Hoggatt's I_{38}). Proof of (*) from the Binet forms $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $L_n = \alpha^n + \beta^n$, $n \in \mathbf{Z}$:

$$\begin{aligned} L_mL_n + 5F_mF_n &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) + 5(\alpha^m - \beta^m)(\alpha^n - \beta^n)/(\sqrt{5}\sqrt{5}) \\ &= 2\alpha^m\alpha^n + 2\beta^m\beta^n = 2(\alpha^{m+n} + \beta^{m+n}) = 2L_{m+n}. \end{aligned}$$

Also solved by Paul S. Bruckman, C. Georghiou, L. Kuipers, John W. Milsom, Andreas N. Philippou, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

Application of the Addition Formula

B-491 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_kF_{n+j} - F_jF_{n+k} = (L_jL_{n+k} - L_kL_{n+j})/5.$$

Solution by J. Suck, Essen, GERMANY

Using (*) in the above solution to B-490, we have

$$\begin{aligned} 5(F_kF_{n+j} - F_jF_{n+k}) &= 2L_{k+n+j} - L_kL_{n+j} - (2L_{j+n+k} - L_jL_{n+k}) \\ &= L_jL_{n+k} - L_kL_{n+j}. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, L. Kuipers, John W. Milsom, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

New Look at Previous Application

B-492 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_n F_{n+j+k} - F_{n+j} F_{n+k} = (L_{n+j} L_{n+k} - L_n L_{n+j+k})/5.$$

Solution by J. Suck, Essen, GERMANY

The same as B-491: rename $k \leftrightarrow n$, $j \to n + j$.

Remark: A companion problem (from Hoggatt's I_{38}) would have been

$$L_k F_{n+j} - L_j F_{n+k} = F_j L_{n+k} - F_k L_{n+j},\ j,\ k,\ n \in \mathbf{Z}.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, John W. Milsom, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

Exponent of 2 in Sum

B-493 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Derive a formula for the largest integer $e = e(n)$ such that 2^e is an integral divisor of

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i}$$

where $\binom{n}{k} = 0$ for $k > n$.

Solution by C. Georghiou, University of Patras, GREECE

Note that, for $n \geq 0$,

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = \frac{1}{2}[(1 + \sqrt{5})^n + (1 - \sqrt{5})^n] = 2^{n-1} L_n.$$

From $2 \nmid L_{3n\pm1}$, $2 | L_{6n}$, $4 \nmid L_{6n}$, $4 | L_{6n+3}$, and $8 \nmid L_{6n+3}$, we get

$$e(n) = \begin{cases} n & \text{if } n \equiv 0 \pmod{6} \\ n+1 & \text{if } n \equiv 3 \pmod{6} \\ n-1 & \text{if } n \not\equiv 0 \pmod{3} \end{cases}$$

Also solved by Paul S. Bruckman, L. Kuipers, Sahib Singh, J. Suck, Charles R. Wall, and the proposer.

Sum of Consecutive Integers

B-494 *Proposed by Philip L. Mana, Albuquerque, NM*

For each positive integer n, find positive integers a_n and b_n such that $101n$ is the following sum of consecutive positive integers:

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n).$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We begin by observing that

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n) = (b_n + 1)a_n + b_n(b_n + 1)/2.$$

Next, we let

$$a_n = 51 - n \text{ and } b_n = 2n - 1 \text{ for each integer } n,\ 1 \leqslant n \leqslant 50$$

and

$$a_n = n - 50 \text{ and } b_n = 100 \text{ for each integer } n,\ n \geqslant 51.$$

Clearly, a_n and b_n are always positive integers. Also,

(1) if $a_n = 51 - n$ and $b_n = 2n - 1$, then

$$\begin{aligned}(b_n + 1)a_n + b_n(b_n + 1)/2 &= (2n)(51 - n) + (2n - 1)n \\ &= 102n - 2n^2 + 2n^2 - n \\ &= 101n;\end{aligned}$$

(2) if $a_n = n - 50$ and $b_n = 100$, then

$$\begin{aligned}(b_n + 1)a_n + b_n(b_n + 1)/2 &= 101(n - 50) + 50(101) \\ &= 101n - 50(101) + 50(101) \\ &= 101n.\end{aligned}$$

Also solved by Ada Booth, Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, H. Klauser & E. Schmutz & M. Wachtel, L. Kuipers, Sahib Singh, J. Suck, and the proposer.

Sum of Consecutive Squares

B-495 *Proposed by Philip L. Mana, Albuquerque, NM*

Characterize an infinite sequence whose first 24 terms are:

1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55,
61, 64, 77, 81, 85, 86, 90, 91, ...

[Note that all perfect squares occur in the sequence.]

Solution by Paul S. Bruckman, Carmichael, CA

The indicated sequence may be characterized as the sequence of positive integers which can be expressed either as squares or as sums of *consecutive* squares, then arranged in increasing order. Equivalently, if the given sequence is denoted by $(x_n)_{n=1}^{\infty}$ and if

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad (\text{with } S_0 \equiv 0),$$

the sequence is characterized as the set of all differences $S_a - S_b$, where $a > b \geqslant 0$, in increasing order.

Also solved by Ada Booth, John W. Milsom, E. Schmutz & M. Wachtel, J. Suck, and the proposer.

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