SOME OPERATIONAL FORMULAS FOR THE q-LAGUERRE POLYNOMIALS

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1. INTRODUCTION

L. B. Rédei [7] proved an operational identity for the Laguerre polynomials that was later generalized by Viskov [9]. Viskov's main results were as follows: if $D = d/dx$, then for $n = 0, 1, 2, \ldots$, we have

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} e^x\{(\alpha + 1 + xD)D\}^n e^{-x} = \frac{(-1)^n}{n!} e^x B^n e^{-x} \tag{1.1}$$

and

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\{(1 + \alpha - x + xD)(1 - D)\}^n \cdot 1, \tag{1.2}$$

where $L^{(\alpha)}(x)$ is the n^{th} Laguerre polynomial.

A third formula of a similar nature was given earlier by Carlitz [2]:

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\prod_{k=1}^{n}(xD - x + \alpha + k) \cdot 1. \tag{1.3}$$

Recently, there has been renewed interest in q-identities and operators as well as in the q-Laguerre polynomial (see, e.g., [3], [5], [6]). Therefore, we felt it would also be interesting to discuss q-generalizations of the identities (1.1)-(1.3). In the following, we shall assume always that $|q| < 1$.

We first introduce the following notation:

$[a]_0 = (a; q)_0 = 1;$

$[a]_n = (a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) \quad (n = 1, 2, 3, \ldots).$

Also, we shall use $[a]_\infty = (a; q)_\infty$ to mean the convergent product

$$\prod_{k=0}^{\infty}(1 - aq^k).$$

It is well known that $[a]_\infty$ is a q-analog of the exponential function. Thus, we have

$$\lim_{q \to 1^-} (-(1 - q)x; q)_\infty^{-1} = e^{-x}.$$

For this reason, the more suggestive notation

$$(x; q)_\infty^{-1} = e_q(x)$$

is used for a q-analog of the exponential function.

The q-derivatives of a function $f(x)$ is given by

$$D_q f(x) = \frac{f(x) - f(xq)}{x},$$

so whenever f has a derivative at x, we have

$$\lim_{q \to 1} \frac{1}{(1 - q)} D_q f(x) = f'(x).$$

We shall also use the substitution operator $\eta: \eta f(x) = f(qx)$. It is related to the q-derivative by means of $xD_q = I - \eta$, where I is the identity operator. Note that x and D_q do not commute.

We recall that the q-Laguerre polynomials [6] are defined by

$$L_n^{(\alpha)}(x|q) = \frac{[q^{\alpha+1}]n}{[q]_n} \sum_{k=0}^{n} \frac{[q^{-n}]\; q^{\frac{1}{2}k(k+1)+k(n+\alpha)}}{[q]_k [q^{\alpha+1}]_k} x^k \tag{1.4}$$

so that

$$\lim_{q \to 1} L_n^{(\alpha)}((1-q)x|q) = L_n^{(\alpha)}(x), \quad n = 0, 1, 2, \ldots .$$

These polynomials, which are orthogonal and belong to an indetermined Stieltjes moment problem ([3] and [6]), were known to W. Hahn [5]. They have, among other properties, a Rodrigues formula:

$$L_n^{(\alpha)}(x|q) = \frac{x^{-\alpha}[-x]_\infty}{[q]_n} D_q^n \left\{ \frac{x^{\alpha+n}}{[-x]_\infty} \right\}. \tag{1.5}$$

Cigler [4] gave the representation

$$L_n^{(\alpha)}(x|q) = \frac{(-1)^n}{[q]_n}(\eta - D_q)^{n+\alpha} x^n = (-1)^n x^{-\alpha} \frac{1}{[q]_n} (\eta - D_q)^n x^{n+\alpha}. \tag{1.6}$$

Representations (1.5) and (1.6) are both of the same nature—the n^{th} iterate of the operator (D_q or $\eta - D_q$, respectively) acts on a function that depends on n also. In some applications, this is a drawback. This is why (1.1) and (1.2) are interesting.

2. A q-ANALOG OF THE REDEI-VISKOV OPERATOR

Put

$$B_q = \{(1 - q^{\alpha+1})I + q^{\alpha+1} x D_q\} D_q. \tag{2.1}$$

Thus, formally, we have

$$\lim_{q \to 1} \frac{B_q}{(1-q)^2} f(x) = (\alpha + 1 + xD)Df(x) = Bf(x),$$

which is the operator that appears in the right-hand side of (1.1).

It is easy to see that

$$B_q x^n = (1 - q^n)(1 - q^{n+\alpha}) x^{n-1}, \tag{2.2}$$

from which we can verify another representation for the B_q operator, namely,

$$B_q = (I - q^{\alpha+1}\eta) D_q \tag{2.3}$$

and

$$B_q = x^{-\alpha} D_q x^{\alpha+1} D_q = D_q x^{1-\alpha} D_q x^\alpha. \tag{2.4}$$

The latter representation shows that the operator B_q is also a q-analog of the Bessel operator (see [10]):

$$B = x^{-\alpha} \frac{d}{dx} x^{\alpha+1} \frac{d}{dx}.$$

From the relation $D_q(1 - q^{\alpha+1}\eta) = (1 - q^{\alpha+2}\eta) D_q$, we get, by induction,

$$B_q^n = \left\{\prod_{k=1}^{n}(1 - q^{\alpha+k}\eta)\right\}D_q^n \quad (n = 0, 1, 2, \ldots). \tag{2.5}$$

It is easy to see that $D_q\{e_q(-x)f(x)\} = e_q(-x)(D_q - \eta)f(x)$. Thus, we have, for any formal power series $F(x)$,

$$F(D_q)\{e_q(-x)f(x)\} = e_q(-x)F(D_q - \eta)f(x). \tag{2.6}$$

Using mathematical induction and noting that

$$(D_q - \eta)(I - q^{\mu}(1 + x)\eta) = (I - q^{\mu+1}(1 + x)\eta)(D_q - \eta),$$

we get

$$\begin{aligned} B_q^n\{e_q(-x)f(x)\} &= e_q(-x)\{(I - q^{\alpha+1}(1 + x)\eta)(D_q - \eta)\}^n \cdot f(x) \\ &= e_q(-x)\left\{\prod_{k=1}^{n}(1 - q^{\alpha+k}(1 + x)\eta)\right\}(D_q - \eta)^n \cdot f(x) \\ &= \frac{1}{x^n}\, e_q(-x)\prod_{k=1}^{n}(1 - q^{\alpha+k-n}(1 + x)\eta)(1 - q^{k-n}(1 + x)\eta) \cdot f(x). \end{aligned}$$

Now, to obtain operational representations for the q-Laguerre polynomials, we first calculate

$$\begin{aligned} B_q^n e_q(-x) &= B_q^n \sum_{k=0}^{\infty} \frac{(-1)^k}{[q]_k}\, x^k \\ &= \sum_{k=n}^{\infty} \frac{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-n+1})(1-q^{k+\alpha})\ldots(1-q^{k-n+\alpha+1})(-x)^{k-n}}{[q]_k} \\ &= (-1)^n[q^{\alpha+1}]_n \sum_{k=0}^{\infty} \frac{[q^{\alpha+n+1}]_k}{[q]_k[q^{\alpha+1}]_k}(-x)^k. \end{aligned}$$

Andrews [1] gave a q-analog of Kummer's Theorem, i.e.,

$$\sum_{k=0}^{\infty} \frac{[\beta]_k[\alpha]_k(-1)^k q^{\frac{1}{2}k(k-1)}}{[q]_k[\gamma]_k[x\alpha]_k}\left(\frac{x\gamma}{\beta}\right)^k = \frac{[x]_\infty}{[x\alpha]_\infty}\sum_{k=0}^{\infty}\frac{[\gamma/\beta]_k[\alpha]_k}{[q]_k[\gamma]_k}\, x^k \tag{2.8}$$

Putting $\alpha = 0$ in this formula, replacing x by $-x$, and then taking $\gamma = q^{\alpha+1}$, $\beta = q^{-n}$, we get that

$$B_q^n e_q(-x) = \frac{(-1)^n[q^{\alpha+1}]_n}{[-x]_\infty}\sum_{k=0}^{n}\frac{[q^{-n}]_k q^{\frac{1}{2}k(k-1)+k(\alpha+n+1)}}{[q]_k}\, x^k = \frac{(-1)^n[q]_n}{[-x]_\infty}\, L_n^{(\alpha)}(x|q).$$

Together with (2.7), this formula gives the following three representations:

$$\begin{aligned} L_n^{(\alpha)}(x|q) &= \frac{(-1)^n}{[q]_n}\{e_q(-x)\}^{-1}B_q^n\{e_q(-x)\}; &(2.9)\\ &= \frac{1}{[q]_n}\prod_{k=1}^{n}(I - q^{\alpha+k}(1 + x)\eta) \cdot 1; &(2.10)\\ &= \frac{(-1)^n}{[q]_n}\{(I - q^{\alpha+1}(1 + x)\eta)(D_q - \eta)\}^n \cdot 1. &(2.11) \end{aligned}$$

If we let $q \to 1$, then (2.9), after suitable normalization by $(1 - q)$, reduces to (1.1); (2.11) reduces to (1.2); and (2.10) reduces to Carlitz's formula (1.3).

Using (2.2) and (1.4), we get, for $m = 0, 1, 2, \ldots,$

$$B_q^m L_n^{(\alpha)}(x|q) = (-1)^m \frac{[q^{\alpha+1}]_n}{[q^{\alpha+1}]_{n-m}} q^{m(m+\alpha)} L_{n-m}^{(\alpha)}(q^{2m}x|q). \tag{2.12}$$

Notice that the operation on $L_n^{(\alpha)}(x|q)$ by B_q reduces the degree by one without changing the value of the parameter α.

There is another q-analog of the exponential function e^{-x}, namely,

$$E_q(x) = \sum_{k=0}^{\infty} \frac{(-1)\ q^{\frac{1}{2}k(k-1)}}{[q]_k} x^k = \prod_{j=0}^{\infty} (1 - xq^j).$$

If we repeat the above calculation, we can show that

$$B_q^n E_q(x) = (-1)^n [q^{\alpha+1}]_n q^{\frac{1}{2}n(n-1)} \sum_{j=0}^{n} \frac{(-1)^j q^{\frac{1}{2}j(j-1)+nj}[q^{\alpha+n+1}]_j}{[q]_j [q^{\alpha+1}]_j} x^j.$$

Once again we can transform the right-hand side of this formula by using (2.8) (with $\alpha = 0$, $\beta = q^{\alpha+n+1}$, $\gamma = q^{\alpha+1}$, and $x \to xq^{2n}$), to obtain

$$B_q^n\{E_q(x)\} = (-1)^n q^{\frac{1}{2}n(n-1)+\alpha n}[q]_n E_q(xq^{2n}) L_n^{(\alpha)}(-xq^{2n-1}|q^{-1}). \tag{2.13}$$

Comparison formulas to this are:

$$e_q(-\eta^{-2}B_q)\{x^n\} = (-1)^n q^{-n(n-1)}[q]_n L_n^{(\alpha)}(q^{\alpha+1}x|q) \tag{2.14}$$

and

$$E_q(-B_q)\{x^n\} = (-1)^n [q]_n q^{\frac{1}{2}n(n-1)} L_n^{(\alpha)}(xq^{2n-1}|q^{-1}). \tag{2.15}$$

Both formulas (2.14) and (2.15) reduce in the case $q \to 1$ to the new formula for the ordinary Laguerre polynomials:

$$e^{-B}x^n = (-1)^n n! L_n^{(\alpha)}(x) \qquad (n = 0, 1, 2, \ldots). \tag{2.16}$$

On the other hand, formula (2.13) reduces to (1.1).

If we calculate the right-hand side of (2.9) directly, we get

$$\begin{aligned} B_q^n e_q(-x) &= (-1)^n \prod_{k=1}^{n} (1 - q^{\alpha+1}\eta) e_q(-x) \qquad (2.17) \\ &= \sum_{j=0}^{n} (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)+(\alpha+1)j} \eta^j e_q(-x) \\ &= e_q(-x) \sum_{j=0}^{n} (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)+\alpha j} [-x]_j. \end{aligned}$$

The second equality is due to the Euler identity

$$\prod_{k=1}^{n} (1 - q^{k-1}x) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)} x^j,$$

where $\begin{bmatrix} n \\ j \end{bmatrix}$ stands for the q-binomial coefficient, i.e., for 1 if $j = 0$ and for

$$(1 - q)(1 - q^{n-1}) \ldots (1 - q^{n-j+1})/(1 - q)(1 - q^2) \ldots (1 - q^j)$$

if $j \geqslant 1$. Combining (2.9) and (2.17), we obtain another expansion for the q-Laguerre polynomial:

$$L_n^{(\alpha)}(x|q) = \frac{1}{[q]_n} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} [-x]_j q^{\frac{1}{2}j(j+1)+\alpha j}. \tag{2.18}$$

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