THE n-NUMBER GAME

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INTRODUCTION AND BACKGROUND

Given an ordered set of n nonnegative integers,

$$S_1 = (a, b, \ldots, m, n),$$

define a linear transformation $T: S_1 \to S_2$, where

$$S_2 = (|a - b|, |b - c|, \ldots, |m - n|, |n - a|).$$

Upon iteration of this transformation, a sequence of n-tuples of nonnegative integers is created. This sequence is called "the n-number game."

The n-number game has been considered primarily in the form of the four-number game. For example,

$$\begin{aligned} S_1 &= (37, 17, 97, 28), \\ S_2 &= (20, 80, 69, 9), \\ S_3 &= (60, 11, 60, 11), \\ S_4 &= (49, 49, 49, 49), \\ S_5 &= (0, 0, 0, 0). \end{aligned}$$

It is well known that the 2^m-number game will always terminate (i.e., reach a 2^m-tuple of all zeros) [7], [10].

The domain of the elements of the n-number game can be extended to the reals with some interesting consequences (e.g., see [3], [6]). However, such extensions will not be dealt with in this paper.

Let us note three well-known properties of the n-number game and make a definition.

(1) There exists a positive integer k such that S_k is an n-tuple all of elements are 0's and α's (e.g., see [1]).

Using Property (1), we will make the following definition.

Definition: The length of the sequence beginning with S_1, $|S_1|$, is $m - 1$, where m is the smallest integer such that the elements of S_{m+k} are all 0's and α's for $k \geqslant 0$.

If $\alpha \neq 0$, we will say that the sequence cycles.

(2) If $S_1 = (ad, bd, \ldots, nd) = d(a, b, \ldots, n) = dS_1^*$, then $|S_1| = |S_1^*|$. (This is easily proven.)

(3) The necessary and sufficient condition for a "parent" to exist for a given n-tuple S_1 is that S_1 can be partitioned into two subsets where the sum of the elements in each subset is the same (e.g., see [1]).

Property (3) implies that, with the exception of the trivial case, the odd-number game will always cycle. To see this, assume that the odd-number game terminates and work backward. If $\beta \neq 0$, a simple parity argument shows that $(\beta, \beta, \ldots, \beta)$ cannot have a parent if n is odd.

An n-tuple will be called an "orphan" if it has no parent. From Property (3), if S_1 is not an orphan, no permutation of S_1 is one; the converse is also true. In the example, $S_1 = (37, 17, 97, 28)$ is an orphan. If S_1 is not an orphan, a parent that is not an orphan is not necessarily unique. An example of this is the case where $S_1 = (5, 3, 8)$, $S_0 = (13, 8, 5)$ or $(3, 8, 11)$.

With these observations in mind, let us proceed to a more systematic study of the three-number game.

THE THREE-NUMBER GAME

In the study of the three-number game, we will use the convention that the first triple in any three-number sequence, $S_1 = (a, b, c)$ is not an orphan and and that $a \geqslant b \geqslant c$. However, by Property (3), $a = b + c$. Therefore,

$$S_1 = (b + c, b, c).$$

Each triple in the cycle of the three-number game is of the form $(0, d, d)$. Since the order of the elements in each triple is of no consequence, no distinction will be made between permutations of a given triple.

Theorem 1: If $S \ = (0, d, d)$, d is the greatest common divisor (g.c.d.) of the elements of S_1 $(d \neq 0)$.

Proof: Let d be the g.c.d. of the elements of $S_1 = (b + c, b, c)$. Let

$$S_1^* = (1/d)S_1 = ((b + c)/d, b/d, c/d) = (b^* + c^*, b^*, d^*).$$

Then $(b^*, c^*) = 1$ and

$$S_2^* = (c^*, b^* - c^*, b^*),$$

where the g.c.d. of the elements of S_2^* is also 1. By induction, the g.c.d. of the elements of S_m^* is 1 for all $m \geqslant 1$. Therefore, if $S_k^* = (0, d^*, d^*)$, $d^* = 1$ (since $d^* \neq 0$ by assumption) and $S_k = (0, d, d)$.

THE LENGTH OF THE THREE-NUMBER GAME

Let us consider an example of the three-number game:

$$\begin{aligned} S_1 &= (17, 37, 20), & S_6 &= (3, 8, 5),\\ S_2 &= (20, 17, 3), & S_7 &= (5, 3, 2),\\ S_3 &= (3, 14, 17), & S_8 &= (2, 1, 3),\\ S_4 &= (11, 3, 14), & S_9 &= (1, 2, 1),\\ S_5 &= (8, 11, 3) & S_{10} &= (1, 1, 0). \end{aligned}$$

The number of appearances of $c = 17$ is three. The generalization of this observation is in Theorem 2.

Theorem 2: If $S_1 = (b + c, b, c)$, where $b = qc + r$, $0 < r < c$, the number of appearances of c is $q + 2 = [b/c] + 2$, where [ ] is the greatest integer function.

Proof: We have

$$\begin{aligned} S_1 &= ((q + 1)c + r, qc + r, c),\\ S_2 &= (qc + r, (q - 1)c + r, c), \end{aligned}$$

and, by simple induction,

$$S_q = (2c + r,\ c + r,\ c).$$

Thus,

$$S_{q+1} = (c + r,\ c,\ r),\ S_{q+2} = (c,\ r,\ c - r),\ \text{and}\ S_{q+3} = (r,\ c - r,\ |c - 2r|).$$

All the elements of S_{q+3} are less than c and, since the transformation T cannot make $\max(S_{k+1}) > \max(S_k)$, the number of appearances of c is $q + 2$.

Note that c is not the g.c.d. of the elements of S_1 in Theorem 2 by the assumption that $0 < r < c$. That case is dealt with in Theorem 3.

Theorem 3: If $S_1 = (b + c,\ b,\ c)$, where $b = qc$, then $|S_1| = q = \dfrac{b}{c}$ $(c \neq 0)$.

Proof: $S_1 = ((q + 1)c,\ qc,\ c)$ and simple induction gives $S_q = (2c,\ c,\ c)$. Thus, $S_{q+1} = (c,\ c,\ 0)$ and $|S_1| = q$.

With these two results, we have Theorem 4, which gives the length of the three-number game and clarifies what is indeed occurring in the sequence.

Theorem 4: If $S_1 = (b + c,\ b,\ c)$, $c \neq 0$, then

$$|S_1| = \sum_{i=1}^{k} q_i,$$

where the q_i, $1 \leqslant i \leqslant k$, are all the quotients in the Euclidean Algorithm for b and c.

Proof: Let $b = q_1 c + r_1$, $0 < r_1 < c$. By Theorem 2,

$$S_{q_1+1} = (c + r_1,\ c,\ r_1).$$

Let $c = q_2 r_1 + r_2$, $0 < r_2 < r_1$, and repeat this process in the style of the Euclidean Algorithm until $r_k = 0$. If

$$t = 1 + \sum_{i=1}^{k-1} q_i,$$

then

$$S_t = (r_{k-1} + r_{k-2},\ r_{k-2},\ r_{k-1}),$$

where $r_{k-2} = q_k r_{k-1}$. By Theorem 3, we have the desired result.

From Theorem 4, we see that the length of the three-number game is greater than or equal to the length of the corresponding Euclidean Algorithm (with equality if and only if all the quotients in the Euclidean Algorithm are ones, i.e., if and only if $b = c$).

There is a special case of the three-number game where the length is very easy to calculate. This case is given in Theorem 5.

Theorem 5: If $S_1 = (b + c,\ b,\ c)$ and d is the g.c.d. of the elements of S_1 and $b \equiv d \pmod{c}$, then

$$|S_1| = \frac{b - d}{c} + \frac{c}{d}.$$

Proof: If $b - kc = d$, Theorem 2 gives

$$S_{k+1} = (b - kc,\ b - (k - 1)c,\ c) = (d + c,\ c,\ d).$$

By Theorem 3,

$$|S_{k+1}| = \frac{c}{d}.$$

Thus,

$$|S_1| = k + \frac{c}{d} = \frac{b - d}{c} + \frac{c}{d}.$$

REMARKS

It is important to note that although all the theorems in this paper refer to S_1, they can be applied to any suitable triple in a sequence by neglecting previous triples.

The three-number game affords a method for finding the g.c.d. of two positive integers b and c [using $S_1 = (b + c, b, c)$ and finding d]. By Theorem 4, the length of the algorithm is small (relative to the size of b and c) if b and c are Fibonacci numbers, while the length of the corresponding Euclidean Algorithm is maximized. In this case, the three-number game takes one more step. In general, however, the three-number game is not a viable method for finding the g.c.d. For a computer that can only add or subtract, it might be useful.

It is known that the length of the four-number game is nearly maximized if the initial entrants are Tribonacci numbers [9]. Can we define "$(n-1)$onacci" numbers that strongly influence the length of the n-number game?

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