FIBONACCI k-SEQUENCES, PASCAL-T TRIANGLES, AND k-IN-A-ROW PROBLEMS

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1. INTRODUCTION AND SUMMARY

In what follows, we use the Fibonacci sequences of order k, as for example in Philippou and Muwafi [2] (although modified somewhat here), and the Pascal-T triangles, as in Turner [6], to solve a number of enumeration problems involving the number of binary numbers of length n which have (or do not have) a string of k consecutive ones, subject to various auxilliary conditions (no k consecutive ones, exactly k, at least k, and so on). Collectively, these kinds of problems might be labelled k-in-a-row problems, and they have a number of interpretations and applications (a few of which are discussed in §4): combinatorics (ménage problems), statistics (runs problems), probability (reliability theory), number theory (compositions with specified largest part). Generating functions, inclusion-exclusion arguments, and the like, are perhaps most commonly used in these problems, but the methods developed here are simple, surprisingly effective, and computationally efficient. Finally, we note that although the string length n is fixed here, some of our results will also apply to parts of [2], [5], [6] (cf., e.g., the Corollary to Theorem 3.1), which discuss the problem of waiting for the k^{th} consecutive success, since the situation there is in some respects essentially that of having a fixed string length of size $n + k$.

Definitions and constructions are in §2, the enumeration theorems are in §3, and §4 gives several examples of their use.

2. MODIFIED k-SEQUENCES, AND TRIANGLES T

We need a slightly altered version of the usual definition of a Fibonacci sequence of order k, one that omits the leading 0, 1.

Definition 2.1: The sequence $\{f_k(n)\}_{n=0}^{\infty}$, $k \geqslant 0$, is said to be the modified Fibonacci sequence of order k if $f_0(n) \equiv 0$, $f_1(n) \equiv 1$, and for $k \geqslant 2$,

$$f_k(n) = \begin{cases} 2^n, & 0 \leqslant n \leqslant k-1 \\ \sum_{i=n-k}^{n-1} f_k(i), & n \geqslant k \end{cases} . \tag{2.1}$$

It will prove convenient to have a notation for the corresponding Pascal-T triangles of order k.

Definition 2.2: For any $k \geqslant 0$, T_k is the array whose rows are indexed by $N = 0, 1, 2, \ldots$, and columns by $K = 0, 1, 2, \ldots$, and whose entries are obtained as follows:

a) T_0 is the all-zero array;

b) T_1 is the array all of whose rows consist of a one followed by zeros;

c) T_k, $k \geqslant 2$, is the array whose $N = 0$ row is a one followed by zeros, whose $N = 1$ row is k ones followed by zeros, and any of whose entries in subsequent rows is the sum of the k entries just above and to the left in the preceding row (with zeros making up any shortage near the left-hand edge).

Definition 2.3: In T_k, denote the entry at the intersection of row N and column K by $C_k(N, K)$.

T_2 is of course *the* Pascal triangle, and we will denote its entries by $\binom{N}{K}$. We note that the T_k can be tabulated for moderate values of N, K and can be considered as available as a binomial table. For $k > 0$, by construction there are $(N(k - 1) + 1)$ nonzero entries in each row, the symmetry relation among the C_k is

$$C_k(N, K) = C_k(N, N(k - 1) - K), \quad 0 \leqslant K \leqslant N(k - 1), \tag{2.2}$$

and the relation among the C_k in adjacent rows is

$$C_k(N, K) = \sum_{j=0}^{k-1} C_k(N - 1, K - j): \tag{2.3}$$

here N, K, and k are nonnegative, an empty sum is taken to be zero, and any C_k with either argument negative is zero. That is, (2.3) just expresses property (c) of the definition of T_k. Also by construction, the relation between the f_k and the C_k is

$$f_k(n) = \sum_{j=0}^{n} C_k(n - j + 1, j), \tag{2.4}$$

so that the $f_k(n)$ are also given by the successive southwest-northeast diagonals of T_k [starting with the (1, 0) entry]. This follows from the recurrence in the definition of $f_k(n)$ and that fact that, by (2.3), each element in the diagonal making up $f_k(n)$ is a sum of k preceding elements.

Definition 2.4: Denote by β_{pk} the number of binary numbers of length n which have a total of p ones and a longest string of exactly k consecutive ones. For any $k \geqslant 2$, define the B_k-array to be

$$\begin{array}{llll}
\beta_{kk} & & & \\
\beta_{k+1,k} & \beta_{k+1,k+1} & & \\
\vdots & \vdots & & \\
\beta_{n,k} & \beta_{n,k+1} & \cdots & \beta_{nn}
\end{array}$$

in which the row elements are associated with a fixed total number of ones, and the column elements with a fixed number of consecutive ones.

3. ENUMERATION THEOREMS

Theorem 3.1: The number of binary numbers of length n which have no k consecutive ones is given by $f_k(n)$, $n \geqslant 0$.

Proof: Let $g(n)$ enumerate the numbers having the property stated. Then we have, schematically.

$$\underbrace{\boxed{\;n\;}}_{g(n)} = \underbrace{\boxed{n-1}}_{g(n-1)}\boxed{0} + \underbrace{\boxed{n-2}}_{g(n-2)}\boxed{01} + \cdots + \underbrace{\boxed{n-k}}_{g(n-k)}\boxed{011\ldots1}\,,$$

with $g(n)$ satisfying the same initial conditions as in (2.1), and so $g(n)$ is just $f_k(n)$.

Corollary 3.1: The number of binary numbers of length n which end with k consecutive ones but have no other string of k consecutive ones is given by

$$f_k(n - k - 1),\ n \geqslant k,$$

and with $f_k(-1) = 1$.

Proof: We know that $f_k(n - 1)$ enumerates the binary strings of length $(n - 1)$ with no k consecutive ones. These, however, form (with one zero at the end) the "first half" of the strings of length n. Thus, passing from $f_k(n)$ to $f_k(n - 1)$ amounts to stripping the last one from the strings in the "last half" of the strings of length n. Continuing the argument in this way, we come to $f_k(n - k - 1)$, which enumerates the strings of length $n - k$ that end with a zero. But, when a string of k consecutive ones is appended, these are precisely the configurations we wish to count.

Remark 3.1: These two results can also be obtained from the work of Philippou and Muwafi [2]. For $k \geqslant 2$, our $f_k(n)$ is their sequence $f_{n+2}^{(k)}$, $n \geqslant 0$. Then, Theorem 3.1 follows from the results in [2], since their $a_n^{(k)}$ is

$$a_n^{(k)} = a_{n+1}^{(k)}(f) = A_{n+1}^{(k)} = f_{n+2}^{(k)} = f_k(n),\ n \geqslant 0,$$

and Corollary 3.1 is equivalent to their Lemma 2.2.

Theorem 3.2: The number of binary numbers of length n which have a longest string of exactly k consecutive ones is given by $f_{k+1}(n) - f_k(n)$, $n \geqslant 1$.

Proof: By Theorem 3.1, $(2^n - f_k(n))$ is the number of configurations with k or more consecutive ones; $(2^n - f_{k+1}(n))$ is the number with $(k + 1)$ or more consecutive ones. Their difference is the number with exactly k.

Corollary 3.2: The column sums of the B_k-array are given by the numbers

$$f_{k+1}(n) - f_k(n).$$

Theorem 3.3: The number of binary numbers of length n that have a total of j ones, no k consecutive is given by $C_k(n - j + 1, j)$.

Proof: Let $g_k(n, j)$ enumerate these numbers. For $0 \leqslant j \leqslant k - 1$, we have, by definition, and because we are in T_k,

$$g_k(n, j) = \binom{n}{j} = C_k(n - j + 1, j),$$

and for $n \geqslant k$, $g_k(n, n) = C_k(1, n) = 0$. Now let $k \leqslant j \leqslant n$. The numbers we want that end in 0 are enumerated by

$$g_k(n - 1, j);$$

those that end in 01 are enumerated by

$$g_k(n - 2, j - 1);$$

those that end in 011 are enumerated by

$$g_k(n - 3, j - 2), \ldots; \text{ and so on.}$$

Then we have the recurrence

$$g_k(n, j) = g_k(n - 1, j) + g_k(n - 2, j - 1) + \cdots + g_k(n - k, j - k + 1). \qquad (*)$$

The conclusion can be proved by induction: the hypothesis asserts that

$$g_k(n - 1, j) = C_k(n - j, j),\ g_k(n - 2, j - 1) = C_k(n - j, j - 1), \ldots,$$
$$g_k(n - k, j - k + 1) = C_k(n - j, j - k + 1).$$

But this implies

$$\begin{aligned} g_k(n, j) &= C_k(n - j, j) + C_k(n - j, j - 1) + \cdots + C_k(n - j, j - k + 1), \text{ by } (*) \\ &= C_k(n - j + 1, j), \text{ by } (2.3). \end{aligned}$$

Corollary 3.3: The number of binary numbers of length n that have a total of j ones and a string of ones of length at least k is given by

$$\binom{n}{j} - C_k(n - j + 1, j).$$

Corollary 3.4: The row sums of the B_k-array are given by the formula of Corollary 3.3.

Theorem 3.4: The columns of the B_k-array (the elements β_{pk} that give the number of binary numbers of length n with a total of p ones and a longest string of exactly k consecutive ones) are given by:

$$\begin{aligned}
\beta_{jj} &= \binom{n}{j} && - C_j(n - (j - 1), j) \\
\beta_{j+1,\,j} &= C_{j+1}(n - j, j + 1) && - C_j(n - j, j + 1) \\
\beta_{j+2,\,j} &= C_{j+1}(n - (j + 1), j + 2) && - C_j(n - (j + 1), j) \qquad 2 \leqslant k \leqslant j \leqslant n \\
&\ \vdots && \ \vdots \\
\beta_{n-1,\,j} &= C_{j+1}(2, n - 1) && - C_j(2, n - 1) \\
\beta_{n,\,j} &= C_{j+1}(1, n) && - C_j(1, n)
\end{aligned}$$

Proof: Having the row sums of B_k for any k by Corollary 3.4, we can obtain B_k column by column.

For completeness, we mention that although B_k was initially defined for $k \geqslant 2$, B_0 and B_1 can also be formed. The $k = 0$ column is a one followed by zeros, and the $k = 1$ column consists of the numbers $\binom{n-p+1}{p}$, $1 \leqslant p \leqslant n$. The corresponding column sums are $f_1(n) - f_0(n) = 1$ and $f_2(n) - f_1(1)$, and the row sums in this case are just the $\binom{n}{p}$.

4. APPLICATIONS

In this section we give three brief examples that are quite straightforward but nevertheless give some idea of the variety of possible interpretations and uses of the previous material.

Example 4.1: Given n objects arranged in a row, the number of ways of choosing j objects from among the n such that among the j chosen no k are consecutive is, by Theorem 3.3, $C_k(n - j + 1, j)$. For $k = 2$, this is just $\binom{n-j+1}{j}$, a result which is one of the principal steps in the solution of the ménage problem [4, p. 33].

Example 4.2: Engineers often increase the reliability of a system by making the conditions under which it fails more stringent. An example from reliability theory is what is called a "consecutive-k-out-of-n:F system" [1]. This is a system of n independent, linearly ordered components, each of which operates (fails) with probability $p(q)$, such that the system fails when and only when k consecutive components fail. What needs to be calculated is the system failure probability, $P_f(n, k)$. If we let a one stand for a failure, then by Corollary 3.3, if we put

$$r_j = \binom{n}{j} - C_k(n - j + 1, j)$$

(j total 1's and at least k consecutive 1's), the failure probability is given by

$$P_f(n, k) = \sum_{j=k}^{n} r_j^{(k)} p^{n-j} q^j.$$

Example 4.3: In number theory, an ordered partition of n is called a composition of n. Let $a(n, k)$ denote the number of compositions of n in which the largest part equals k. There is a natural one-to-one correspondence between the compositions $a(n, k)$ and the number of binary numbers of length n beginning with a zero, and containing the pattern 1...1 with $k - 1$ ones but not the pattern 1...1 with k ones; that is, any integer m in the composition is represented by the pattern 01...1 with $m - 1$ ones. But if the string of length n must begin with a zero, we are just considering the "first half" of all the strings of length n. This is equivalent to considering strings of length $n - 1$ that have a largest consecutive-ones substring of length $k - 1$, and so Theorem 3.2 solves the problem of enumerating the $a(n, k)$; i.e., $a(n, k) = f_k(n - 1) - f_{k-1}(n - 1)$, $n \geqslant 1$, $1 \leqslant k \leqslant n$. A short table follows:

$a(n, k)$:

$k \backslash n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2		1	2	4	7	12
3			1	2	5	11
4				1	2	5
5					1	2
6						1

(For a generating function approach to this enumeration, see John Riordan [3, Ch. 6].)

It seems fair to say that the generalized Fibonacci-sequence/Pascal-triangle approach, as well as being interesting in its own right, is quite useful and a reasonable alternative to the generating function or multinomial methods often used in these kinds of problems.

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