A NOTE ON APERY NUMBERS

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To prove the irrationality of the number

$$\zeta(3) = \sum_{n=1}^{\infty} (1/n^3),$$

Apèry recently introduced the sequence $\{a_n, n \geqslant 0\}$ defined by the recurrence relation

$$a_0 = 1, \; a_1 = 5,$$

and

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3 a_{n-2} = 0 \tag{1}$$

for $n \geqslant 2$. Apèry proved that for the pair $(a_0, a_1) = (1, 5)$, all the a_n's are integers, and each a_n has the representation

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The first six a_n's are:

$$a_0 = 1, \; a_1 = 5, \; a_2 = 73, \; a_3 = 1445, \; a_4 = 33001, \; a_5 = 819005$$

(see [1]).

Some congruence properties of Apèry numbers are established in [1] and [2]. In [1], it is asked if there are values for the pair (a_0, a_1) other than (1, 5) in (1) that would produce a sequence $\{a_n, n \geqslant 0\}$ of integers. In particular, taking $a_0 = 1$, it is also asked if there is a necessary and sufficient condition on a_1 for all the a_n's to be integers. In answering these questions, we first prove the following theorem.

Theorem: Let $a_0 = 0$. The condition $a_1 = 0$ is necessary and sufficient for all of the a_n's defined by Apèry recurrence relation to be integers.

Proof: The sufficiency is clear. To prove the necessity we assume, on the contrary, that there exists an integer $k \neq 0$ such that all of the b_n's produced by Apèry recurrence relation with $b_0 = 0$, $b_1 = k$ are integers. Without loss of generality, we assume $k > 0$.

For the sequence $\{b\ , n \geqslant 0\}$, (1) can be written as

$$n^3 b_n = (34n^3 - 51n^2 + 27n - 5)b_{n-1} - (n-1)^3 b_{n-2} \tag{2}$$

and hence

$$b_n - b_{n-1} = \left(33 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right)b_{n-1} - \left(1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3}\right)b_{n-2}.$$

Since we have

$$\left(33 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right) - \left(1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3}\right) = 4\left(2 - \frac{1}{n}\right)^3 > 0$$

for all $n \geqslant 2$, it follows that $b_{n-1} > b_{n-2} \geqslant 0$ implies $b_n > b_{n-1}$. Since $b_1 = k > 0 = b_0$, then, by induction, $b_n > b_{n-1}$ for all $n \geqslant 1$. Similarly, since $a_1 =$

$5 > 1 = a_0$, we also have $a_n > a_{n-1}$ for all $n \geqslant 1$. Thus, $a_n > 0$ and $b_n > 0$ for $n \geqslant 1$.

The equation (2), with $n = 2$, implies that $8b_2 = 117b_1$. Therefore, we have $b_1/a_1 < b_2/a_2$. Now we prove that for each integer $n \geqslant 2$,

$$\frac{b_{n-1}}{a_{n-1}} < \frac{b_n}{a_n}.$$

The Apèry recurrence relation (1) can be written as

$$(34n^3 - 51n^2 + 27n - 5)a_{n-1} = n^3a_n + (n - 1)^3a_{n-2}. \tag{4}$$

Let $\lambda_i = b_i/a_i$, $i \geqslant 1$. If $\lambda_{n-2} < \lambda_n$, then from (4) we have

$$\begin{aligned}(34n^3 - 51n^2 + 27n - 5)\lambda_{n-1}a_{n-1} &= n^3\lambda_n a_n + (n - 1)^3\lambda_{n-2}a_{n-2}\\ &< \lambda_n(n^3a_n + (n - 1)^3a_{n-2}),\end{aligned}$$

and

$$(34n^3 - 51n^2 + 27n - 5)\lambda_{n-1}a_{n-1} > \lambda_{n-2}(n^3a_n + (n - 1)^3a_{n-2}).$$

Hence,

$$\lambda_{n-2} < \lambda_n \text{ implies } \lambda_{n-2} < \lambda_{n-1} < \lambda_n.$$

Similarly,

$$\lambda_{n-2} \geqslant \lambda_n \text{ implies } \lambda_{n-2} \geqslant \lambda_{n-1} \geqslant \lambda_n.$$

Therefore, the inequality $\lambda_{n-2} < \lambda_{n-1}$ implies $\lambda_{n-1} < \lambda_n$. Now since (3) holds for $n = 2$, it also holds for all $n \geqslant 2$.

From (4), we get

$$\frac{b_n}{a_n} = \frac{(n + 1)^3b_{n-1} + n^3b_{n-1}}{(n + 1)^3a_{n-1} + n^3a_{n-1}}.$$

Hence, clearing the denominator and collecting terms yields

$$(3n^2 + 3n + 1)\left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}\right)a_na_{n+1} = n^3((a_{n-1}b_n - b_{n-1}a_n) - (a_nb_{n+1} - b_na_{n+1})).$$

Thus, using (3), we get $a_{n-1}b_n - b_{n-1}a_n > a_nb_{n+1} - b_na_{n+1}$ for all $n \geqslant 2$; hence,

$$a_nb_{n+1} - b_na_{n+1} \leqslant (a_{n-1}b_n - b_{n-1}a_n) - 1 \tag{5}$$

for all $n \geqslant 2$. Note that (3) also implies

$$a_nb_{n+1} - b_na_{n+1} > 0 \tag{6}$$

for all $n \geqslant 2$. Comparing (5) and (6), we can clearly see a contradiction. This completes the proof. ■

We have the following corollary as a consequence of the above theorem.

Corollary: It is necessary and sufficient that the pair $(a_0, a_1) = c(1, 5)$, where c is any integer, for all the a_n's in (1) to be integers.

Proof: The sufficiency follows immediately from the linearity of the relation (1) relative to a_n's. To prove the necessity, suppose $(a_0, a_1) = (c, d)$ is a pair that causes all of the a_n's to be integers. By the linearity of (1), the pair $(0, d - 5c) = (c, d) - c(1, 5)$ is also a pair that causes all of the a_n's to be integers. By the theorem, $d - 5c = 0$, that is, $(c, d) = c(1, 5)$.

As a last comment, we slightly improve a lemma presented in [2].

By multiplying $(6n^2 - 3n + 1)$ to the equation

$$(n + 1)^3 a_{n+1} - (34(n + 1)^3 - 51(n + 1)^2 + 27(n + 1) - 5)a_n + n^3 a_{n-1} = 0,$$

we obtain

$$(6n^2 - 3n + 1)((n^3 + 3n^2 + 3n + 1)a_{n+1} - (34n^3 + 51n^2 + 27n + 5)a_n + n^3 a_{n-1}) = 0,$$

and hence,

$$a_{n+1} \equiv (5 + 12n)a_n \pmod{n^3}$$

for $n \geqslant 2$. The same result was given in [2] with (mod n^2) instead of (mod n^3).

REFERENCES

1. S. Chowla, J. Cowles, & M. Cowles. "Congruence Properties of Apèry Numbers." *J. Number Theory* 12 (1980):188-90.
2. J. Cowles. "Some Congruence Properties of Three Well-Known Sequences: Two Notes." *J. Number Theory* 12 (1980):84-86.

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