ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS and SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-372 *Proposed by M. Wachtel, Zürich, Switzerland*

There exist infinitely many sequences, each with infinitely many solutions of the form:

$$\begin{array}{l||lll}
\underline{A}\cdot x_1^2 + C = \underline{B}\cdot y_1^2 & \underline{A} = F_{n+3} & \underline{C} = L_n & \underline{B} = F_{n+1} \\
\underline{A}\cdot x_2^2 + C = \underline{B}\cdot y_2^2 & \underline{x_1} = 1 & & \underline{y_1} = 2 \\
\underline{A}\cdot x_3^2 + C = \underline{B}\cdot y_3^2 & \underline{x_2} = F_{n-1}F_n + F_{n+1}^2 & & \underline{y_2} = 2F_{n+1}^2 \\
\cdots\cdots \quad \cdots \quad \cdots\cdots & & & \\
\underline{A}\cdot x_m^2 + C = \underline{B}\cdot y_m^2 & \underline{x_3} = 2F_{2n+4} + (-1)^n & & \underline{y_3} = 2F_{2n+5}
\end{array}$$

Find a recurrence formula for x_4/y_4, x_5/y_5, ..., x_m/y_m (y_m = dependent on x_m).

Examples: $(x_1 - x_3)$

$$\begin{array}{l||l}
\underline{n = 3} & \text{(in numbers)} \\
\underline{F_6}\cdot(1)^2 \qquad\qquad + \underline{L_3} = \underline{F_4}\cdot(2)^2 & 8\cdot 1 \quad + \underline{4} = \underline{3}\cdot 2^2 \\
\underline{F_6}\cdot(F_2F_3 + F_4^2)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_4^2)^2 & \underline{8}\cdot 11^2 + \underline{4} = \underline{3}\cdot 18^2 \\
\underline{F_6}\cdot(2F_{10} - 1)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_{11})^2 & \underline{8}\cdot 109^2 + \underline{4} = \underline{3}\cdot 178^2 \\
\hline
\underline{n = 4} & \\
\underline{F_7}\cdot(1)^2 \qquad\qquad + \underline{L_4} = \underline{F_5}\cdot(2)^2 & \underline{13}\cdot 1 \quad + \underline{7} = \underline{5}\cdot 2^2 \\
\underline{F_7}\cdot(F_3F_4 + F_5^2)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_5^2)^2 & \underline{13}\cdot 31^2 + \underline{7} = \underline{5}\cdot 50^2 \\
\underline{F_7}\cdot(2F_{12} + 1)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_{13})^2 & \underline{13}\cdot 289^2 + \underline{7} = \underline{5}\cdot 466^2
\end{array}$$

H-373 *Proposed by Andreas N. Philippou, University of Patras, Greece*

For any fixed integers $k \geq 0$ and $r \geq 2$, set

$$f_{n+1,r}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + n_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1}, \quad n \geq 0.$$

Show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell,r-1}^{(k)}, \quad n \geqslant 0.$$

H-374 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

If $\sigma^*(n)$ is the sum of the unitary divisors of n, then

$$\sigma^*(n) = \prod_{p^e \| n} (1 + p^e),$$

where p^e is the highest power of the prime p that divides n. The ratio $\sigma^*(n)/n$ increases as new primes are introduced as factors of n, but decreases as old prime factors appear more often. As N increases, is $\sigma^*(N!)/N!$ bounded or unbounded?

H-375 *Proposed by Piero Filipponi, Rome, Italy*

Conjecture 1

If $F_k \equiv 0 \pmod{k}$ and $k \neq 5^n$, then $k \equiv 0 \pmod{12}$.

Conjecture 2

Let $m > 1$ be odd. Then, $F_{12m} \equiv 0 \pmod{12m}$ implies either 3 divides m or 5 divides m.

Conjecture 3

Let $p > 5$ be a prime such that $p \nmid F_{24}$, then $F_{12m} \not\equiv 0 \pmod{12m}$.

Conjecture 4

If $L_k \equiv 0 \pmod{k}$, then $k \equiv 0 \pmod{6}$ for $k > 1$.

SOLUTIONS

Lotta Sequences

H-350 *Proposed by M. Wachtel, Zürich, Switzerland*
(Vol. 21, no. 1, February 1983)

There exist an infinite number of sequences, each of which has an infinite number of solutions of the form:

$$A \cdot x_1^2 + 1 = 5 \cdot y_1^2 \qquad \underline{A} = 5 \cdot (a^2 + a) + 1 \qquad \underline{a} = 0, 1, 2, 3, \ldots$$

$$A \cdot x_2^2 + 1 = 5 \cdot y_2^2$$

$$A \cdot x_3^2 + 1 = 5 \cdot y_3^2 \qquad \underline{x_1} = 2;\ \underline{x_2} = 40(2a+1)^2 - 2$$

$$\cdots\cdots \quad \cdots \quad \cdots\cdots$$

$$A \cdot x_n^2 + 1 = 5 \cdot y_n^2 \qquad \underline{y_1} = 2a + 1;\ \underline{y_2} = (2a+1) \cdot (16A + 1)$$

Find a recurrence formula for x_3/y_3, x_4/y_4, ..., x_n/y_n (y_n = dependent on x_n).

Examples

$\underline{a = 0}$	$1 \cdot \left(\frac{L_3}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_3}{2}\right)^2$	$\underline{a = 1}$	$11 \cdot 2^2 + 1 = 5 \cdot 3^2$
	$1 \cdot \left(\frac{L_9}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_9}{2}\right)^2$		$11 \cdot 358^2 + 1 = 5 \cdot 531^2$
	$1 \cdot \left(\frac{L_{15}}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_{15}}{2}\right)^2$		$11 \cdot 63722^2 + 1 = 5 \cdot 94515^2$
	$1 \cdot \ldots \quad + 1 = 5 \cdot \ldots$		$11 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 5}$ $151 \cdot 2^2 + 1 = 5 \cdot 11^2$

$151 \cdot 4{,}838^2 + 1 = 5 \cdot 26{,}587^2$

$151 \cdot 11{,}698{,}282^2 + 1 = 5 \cdot 64{,}287{,}355^2$

$151 \quad \ldots \qquad + 1 = 5 \quad \ldots$

Solution by Paul S. Bruckman, Carmichael, CA

The general solution of the Diophantine equation:

$$5y^2 - Ax^2 = 1 \tag{1}$$

is given by

$$x_n = \frac{u^{2n-1} - v^{2n-1}}{2\sqrt{A}}, \quad y_n = \frac{u^{2n-1} + v^{2n-1}}{2\sqrt{5}}, \quad n = 1, 2, \ldots, \tag{2}$$

where

$$u = (2a + 1)\sqrt{5} + 2\sqrt{A}, \quad v = (2a + 1)\sqrt{5} - 2\sqrt{A}. \tag{3}$$

Note that $uv = 5(2a + 1)^2 - 4(5(a^2 + a) + 1) = 1$. Also,

$$u^2 = 5(2a + 1)^2 + 4A + 4(2a + 1)\sqrt{5A} = 8A + 1 + 4(2a + 1)\sqrt{5A},$$

and

$$\begin{aligned} u^4 &= (8A + 1)^2 + 80A(2a + 1)^2 + 8(2a + 1)(8A + 1)\sqrt{5A} \\ &= (8A + 1)^2 + 80A\left(1 + \frac{4}{5}(A - 1)\right) + 2(8A + 1)(u^2 - 8A - 1) \\ &= -(8A + 1)^2 + 16A + 64A^2 + 2(8A + 1)u^2 = 2(8A + 1)u^2 - 1. \end{aligned}$$

Note that v satisfies the same relation. Thus,

$$w^4 - 2Bw^2 + 1 = 0, \tag{4}$$

where w denotes either u or v, and $B = 8A + 1 = 40a^2 + 40a + 9$. From (4), we readily deduce the recursions:

$$z_{n+2} - 2Bz_{n+1} + z_n = 0, \; n = 1, 2, \ldots, \tag{5}$$

where z denotes either x or y.

Now, let

$$r_n = \frac{x_n}{y_n}, \; n = 1, 2, \ldots . \tag{6}$$

Then, using (5), we obtain:

$$r_{n+2} = \frac{2Bx_{n+1} - x_n}{2By_{n+1} - y_n} = \frac{2Br_{n+1} - \dfrac{r_n y_n}{y_{n+1}}}{2B - \dfrac{y_n}{y_{n+1}}} = r_{n+1} + \frac{(r_{n+1} - r_n)y_n}{2By_{n+1} - y_n}.$$

Hence,

$$\frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = 2B \cdot \frac{y_{n+1}}{y_n} - 1,$$

or, equivalently:

$$\frac{y_{n+1}}{y_n} = \frac{1}{2B} \cdot \frac{r_{n+2} - r_n}{r_{n+2} - r_{n+1}}. \tag{7}$$

Also, using (5) and (7),

$$\frac{y_{n+2}}{y_{n+1}} = 2B - \frac{y_n}{y_{n+1}} = \frac{1}{2B} \cdot \frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}},$$

which implies:

$$4B^2 - 4B^2\left(\frac{r_{n+2} - r_{n+1}}{r_{n+2} - r_n}\right) = \frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}},$$

or

$$\frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}} = 4B^2\left(\frac{r_{n+1} - r_n}{r_{n+2} - r_n}\right). \tag{8}$$

Solving for r_{n+3} in (8) yields the desired recursion:

$$r_{n+3} = \frac{r_{n+1}(r_{n+2} - r_n) - 4B^2 r_{n+2}(r_{n+1} - r_n)}{r_{n+2} - r_n - 4B^2(r_{n+1} - r_n)}. \tag{9}$$

Also solved by the proposer.

Hats Off

H-352 *Proposed by Stephen Turner, Babson College, Babson Park, MA (Vol. 21, no. 2, May 1983)*

One night during a national mathematical society convention, n mathematicians decided to gather in a suite at the convention hotel for an "after hours chat." The people in this group share the habit of wearing the same kind of hats, and each brought his hat to the suite. However, the chat was so engaging that at the end of the evening each (being deep in thought and oblivious to the practical side of matters) simply grabbed a hat at random and carried it away by hand to his room.

Use a variation of the Fibonacci sequence for calculating the probability that none of the mathematicians carried his own hat back to his room.

Solution by J. Suck, Essen, Germany

The problem is Montmort's 1708 "problème des rencontres" (see [1], p. 180) and the required solution is an old hat of Euler's. In [2], he proves by an easy-to-find combinatorial argument that the number D_n of derangements (= permutations without fixed point) of $\{1, 2, \ldots, n\}$ satisfies the recurrence

$$D_{n+2} = (n+1)(D_{n+1} + D_n),\ D_0 = 1,\ D_1 = 0,$$

and hence,

$$D_{n+1} = (n+1)D_n + (-1)^{n+1}.$$

We can proceed to show by induction, then, the "closed" expression (also known to Euler [3])

$$D_n = n! \sum_{\nu=0}^{n} (-1)^\nu/\nu!,$$

which has to be divided by $n!$ to get the required probability.

Disclosing that k out of the n gathering mathematicians were members of the Fibonacci Association, one is led to ask a bit more generally for the probability that none at least of that group carried back his own hat. Denote by $D_{n,k}$ the numerator, that is, the number of permutations of $\{1, \ldots, n\}$ with no fixed point in $\{1, \ldots, k\}$. We have

$$D_{n,k} = \sum_{\nu=k}^{n} \binom{n-k}{\nu-k} D_\nu = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} (n-\nu)!$$

where the right-hand side equality was a recent problem by Wang [4]. It pays to look at the triangular array $D_{n,k}$, $0 \leqslant k \leqslant n$. Morgenstern ([5], p. 15, Problem 11) offers the recurrence

$$D_{n,k} = (n-1)D_{n-1,k-1} + (k-1)D_{n-2,k-2}.$$

To get started, note that the first two columns are

$$D_{n,0} = n! \quad \text{and} \quad D_{n,1} = (n-1)!(n-1), \; n \geqslant 1.$$

We may also start from Euler's edge with

$$D_{n,k} = D_{n-1,k} + D_{n,k+1}.$$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	0				
2	2	1	1			
3	6	4	3	2		
4	24	18	14	11	9	
5	120	96	78	64	53	44

Denoting by R_n the n^{th} row sum, we find

$$R_{n+2} = (n+2)(R_{n+1} + R_n) \quad \text{and} \quad R_n = (n+1)! - D_{n+1}.$$

And for the n^{th} rising diagonal sum $S_n := D_{n,0} + D_{n-1,1} + \cdots$, we have

$$S_{n+2} = \begin{cases} S_{n+1} - S_n + (n+2)! & \text{for } n \text{ even,} \\ S_{n+1} + S_n + (n+2)! - D_{(n+1)/2} & \text{for } n \text{ odd.} \end{cases}$$

Peeping into the future, we see n growing, so finally, for our probability

$$\lim_{n\to\infty} D_{n,k}/n! = \begin{cases} 1 & \text{if } k \text{ remains constant,} \\ 1/e & \text{if } n-k \text{ does.} \end{cases}$$

References

1. L. Comtet. *Advanced Combinatorics*. Dordrecht: D. Reidel, 1974.
2. L. Euler. "Solutio quaestionis curiosae ex doctrina combinationum." *Leonhardi Euleri Opera Omnia*, Ser. Prima 7:435-40.
3. L. Euler. "Calcul de la probabilité dans le jeu de rencontre." *Loc. cit.*: 11-25; "Problema de permutationibus." *Loc. cit.*:542-45.
4. E. T. H. Wang. Elementary Problem E 2947. *Amer. Math. Monthly* 89 (1982): 334.
5. D. Morgenstern. *Einführung in die Wahrscheinlichkeitsrechnung und mathematische Statistik*. Berlin: Springer-Verlag, 1964.

Also solved by P. Bruckman, E. Schmutz and M. Wachtel, N. Saxena and Sridhar Manthani (paper), and the proposer.

Dual Solution

H-353 *Proposed by Jerry Metzger, University of North Dakota, Grand Forks, ND (Vol. 21, no. 2, May 1983)*

For a positive integer n, describe all two-element sets $\{a, b\}$ for which there is a polynomial $f(x)$ such that $f(x) \equiv 0 \pmod{n}$ has solution set exactly $\{a, b\}$.

Solution by L. Kuipers, Switzerland

Let the congruence related to a pair (a, b) be written in the form

$$(x - a)(x - b) \equiv 0 \pmod{n}. \tag{1}$$

If a and b are the only solutions of (1), then (a, b) is called an S_n-pair, or S-pair. We assume throughout $a \neq b$, and distinguish several cases:

(i) Let $n = p$, p prime. Then any pair (a, b) forms a set $\{a, b\}$.

(ii) Let $n = p^2$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^2}. \tag{2}$$

Each factor of the left-hand side of (2) if not zero produces at most one factor p. Hence, if $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. If $a \equiv b \pmod{p}$, then $x \equiv a$, $x \equiv b \pmod{p^2}$ are not the only solutions of (2). Let $a - b = kp$ $[k \not\equiv 0 \pmod{p}]$. Then take $x = a + p$, and substitution in (2) gives

$$(x - a)(x - b) \equiv p^2(1 + k) \equiv 0 \pmod{p^2}.$$

(iii) Let $n = p^3$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^3}. \tag{3}$$

If $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. If $a \equiv b \pmod{p^2}$, then (a, b) is an S-pair if and only if $p = 2$. We have here $a = b + p^2$, for in $a - b = kp^2$ we have $k < p$. If $p \geqslant 3$, $a \equiv b \pmod{p^2}$, then there is always a solution of (3) distinct from $x \equiv a$ and $x \equiv b \pmod{p^3}$. Let $a - b = kp^2$, $k < p$. Take $x = a + p^2$, then

$$(x - a)(x - b) \equiv p^2(p^2 + kp^2) \equiv 0 \pmod{p^3}.$$

So (a, b) is not an S-pair.

If $a \equiv b \pmod{p}$, or $a - b = kp$, then, taking $x = a + p^2$, we have to take $k = 1$. In these cases, (a, b) is not an S-pair.

(iv) Let $n = p^4$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^4}. \tag{4}$$

If $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. Let $a \equiv b \pmod{p}$, or $a - b = kp$ $(k < p^3)$. Now take $x = a + p^3$ in (4).

$$(x - a)(x - b) = p^2(p^2 + kp) \equiv 0 \pmod{p^3}.$$

One then obtains $p^3(p^3 + kp) \equiv 0 \pmod{p^4}$.

In general, if $n = p$ $(k \geqslant 5)$, p being a prime, then $a \not\equiv b \pmod{p}$ yields the sets $\{a, b\}$, while $x = a + p^{k-1}$ gives a third solution to the congruence $(x - a)(x - b) \equiv 0 \pmod{p^k}$ if $a - b \equiv 0 \pmod{p}$.

(v) Let $n = pq$; p, q being primes, $(p, q) = 1$. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{pq}. \tag{5}$$

For a solution of (5), one factor of the left-hand side of (5) must produce p and the second one the factor q. So, consider the system

$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv b \pmod{q}. \end{aligned} \tag{6}$$

Let $qb_1 \equiv 1 \pmod{p}$, $pb_2 \equiv 1 \pmod{q}$. Then a solution of (6) is given by

$$x_0 = qb_1a + pb_2b \pmod{pq}.$$

Now $x_0 \equiv a \pmod{pq}$ implies $a \equiv b \pmod{q}$, and conversely. Thus, if $a \equiv b \pmod{p}$, then (a, b) is an S-pair, and if $a \equiv b \pmod{q}$, then (a, b) is an S-pair.

Now, assume $a \not\equiv b \pmod{p}$, $a \not\equiv b \pmod{q}$. There are integers x and y such that $xp + yq = 1$. Hence, $a - b = (a - b)xp + (a - b)yq$ or $a - b = kp + \ell q$ or $a - kp = b + \ell q$. Thus, $a - kp$ is another solution of (5), as can be seen by substitution.

(vi) Let $n = m_1m_2$, $(m_1, m_2) = 1$. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{(m_1m_2)}. \tag{7}$$

For an extra solution of (7), it is sufficient that the first factor of the left-hand side of (7), i.e., $(x - a)$, is a multiple of m_1 and the second one is divisible by m_2. So consider the system

$$\begin{aligned} x &\equiv a \pmod{m_1} \\ x &\equiv b \pmod{m_2}. \end{aligned} \tag{8}$$

Let $m_2b_1 \equiv 1 \pmod{m_1}$, $m_1b_2 \equiv 1 \pmod{m_2}$.

Then a solution of (8) is given by

$$x_0 = m_2b_1a + m_1b_2b \pmod{m_1m_2}.$$

Now, $x_0 \equiv a \pmod{m_1m_2}$ implies $a \equiv b \pmod{m_2}$, and conversely. Also $x_0 \equiv b \pmod{m_1m_2}$ implies $a \equiv b \pmod{m_1}$. Hence, if $a \equiv b \pmod{m_2}$, then (a, b) is an S-pair; if $a \equiv b \pmod{m_2}$, then (a, b) is an S-pair.

Assume now that $m_1 \nmid a - b$ and $m_2 \nmid a - b$. There are integers x and y such that $xm_1 + ym_2 = 1$. Hence,

$$a - b = (a - b)xm_1 + (a - b)ym_2 = km_1 + \ell m_2 \quad \text{or} \quad a - km_1 = b + \ell m_2.$$

Thus, $a - km_1$ is another solution of (7) as follows by substitution.

(vii) After the preceding cases, it is not difficult to deal with the general case

$$n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}.$$

As soon as $a - b$ is divisible by $p_i^{r_i}$, we have an S-pair, and if $p_i^{r_i} \nmid a - b$, $i = 1, 2, \ldots, t$, then there are, besides a and b, extra solutions of the involved congruence.

Also solved by the proposer.

Not for Squares

H-354 *Proposed by Paul Bruckman, Concord, CA (Vol. 21, no. 2, May 1983)*

Find necessary and sufficient conditions so that a solution in relatively prime integers x and y can exist for the Diophantine equation:

$$ax^2 - by^2 = c,$$

given that a, b, and c are pairwise relatively prime positive integers, and moreover, a and b are not both perfect squares.

Solution by M. Wachtel, Zürich, Switzerland

1.1 In conformity with Problem H-350, which represents a special case of H-354, the following symbolism is used:

$A \cdot x_1^2 + C = B \cdot y_1^2$	A, B, and C = constant values
$A \cdot x_2^2 + C = B \cdot y_2^2$	A, B = relatively prime
.....	C = dependent on A and B
$A \cdot x_n^2 + C = B \cdot y_n^2$	C, x, y = reciprocally dependent

1.2 These infinite sequences consist of an undeterminable number of groups and classes. Considering the limited space available, only main fragments of the whole issue can be dealt with here.

2.1 First, we have to determine the desired C and the least x_1 and y_1 for a given A and B.

2.2 As to C, we have to distinguish between:

a) C = 1, 2, a prime, a double prime, or a quadruple prime. Then, only one sequence exists, containing all terms possible.

b) If C is one of the remaining composite numbers, then two or more sequences exist. No term in a sequence is identical to a term in another sequence.

2.3 To determine x_2, y_2, there does not (presumably) exist a general formula, but an undeterminable number of different construction rules, according to the group or class to which the sequence belongs. When both x_1, y_1 and x_2, y_2 are found, all other terms are determined. See Section 3 below.

3.1 For x_3, y_3, x_4, y_4, ..., x_n, y_n, the following procedure leads to a recurrence formula which comprehends the whole of the terms in integers that are possible.

3.2 The following applies if: $A < B$.

3.3 Let: $x_2 - x_1 = \underline{u}$ and $y_2 - y_1 = \underline{v}$.

3.4 Divide $\underline{u}$ and $\underline{v}$ by their greatest common divisor $\underline{d}$ and let:

$$\frac{u}{d} = \underline{U} \quad \text{and} \quad \frac{v}{d} = \underline{V}.$$

$\underline{U}$, $\underline{V}$ = *auxiliary constants* relatively prime.

3.5 Let: $U \cdot y_1 - V \cdot x_1 = \underline{D}$. Now, let

$$\frac{x_1 + x_2}{D} = \underline{F} \quad \text{and} \quad \frac{y_1 + y_2}{D} = \underline{G}.$$

$\underline{F}$, $\underline{G}$ = *auxiliary constants*.

3.6 Further, let:

$$U \cdot y_1 + V \cdot x_1 = S_1$$
$$U \cdot y_2 + V \cdot x_2 = S_2$$
$$\cdots\cdots$$
$$U \cdot y_n + V \cdot x_n = S_n$$

3.7 and we obtain the recurrence formula:

$$x_3 = F \cdot S_1 + x_1 \qquad y_3 = G \cdot S_1 - y_1$$

$$x_4 = F \cdot S_2 + x_2 \qquad y_4 = G \cdot S_2 - y_2$$

$$\ldots \qquad \ldots$$

$$x_n = F \cdot S_{n-2} + x_{n-2} \qquad y_n = G \cdot S_{n-2} - y_{n-2}$$

3.8 The auxiliary constants U, V and F, G hold also for any C in a sequence corresponding to A, B. That means it suffices to choose an arbitrary C (fitted to A, B) to determine U, V and F, G for any sequence A, B.

3.9 If $A > B$, the procedure is similar to that of 3.2 but is omitted here to conserve space.

3.10 Examples (for the sake of brevity and lucidity, the constants A, B, and C are listed only once, and the power "2" above x and y is omitted throughout).

3.11 Example I: $A = 21$, $B = 31$, $C = 19$ (= prime, one sequence only, see 2.2a).

21 • 6 (x_1) $u = 124$ $d = 2$ (see 3.4) $v = 102$ 31 • 5 (y_1)

130 (x_2)   107 (y_2)

$\frac{u}{d} = U$ $U \cdot y_1 - V \cdot x_1 = D$ $\frac{v}{d} = V$

$\frac{124}{2} = 62$ $62 \cdot 5 - 51 \cdot 6 = 4$ $\frac{102}{2} = 51$

$\frac{x_1 + x_2}{D} = F$ $\frac{y_1 + y_2}{D} = G$

$\frac{136}{4} = 34$ $\frac{112}{4} = 28$

20,950 (x_3)	$U \cdot y_1 + V \cdot x_1 = S_1$	17,243 (y_3)
$= F \cdot S_1 + x_1$		$= G \cdot S_1 - y_1$
$= 34 \cdot 616 + 6$	62 5 + 51 6 = 616	$= 28 \cdot 616 - 5$
451,106 (x_4)	$U \cdot y_2 + V \cdot x_2 = S_2$	371,285 (y_4)
$= F \cdot S_2 + x_2$		$= G \cdot S_2 - y_2$
$= 34 \cdot 13{,}264 + 130$	62 107 + 51 130 = 13,264	$= 28 \cdot 13{,}264 - 107$
72,696,494 (x_5)		59,833,205 (y_5)

Example II: $A = 21$, $B = 31$, $C = 82$ (see 2.2a)

21 • 23 (x_1)	31 • 19
147 (x_2)	121
79,957 (x_3)	65,809
510,113 (x_4)	419,851

3.12 Example III: $A = 6$, $B = 41$, $C = 1001$ (= composite number = $7 \cdot 11 \cdot 13$ yields four different sequences, see 2.2b), x_1, x_2, x_3.

Sequence (a): x_1, x_2, x_3

6 •	2	41 •	5
	983,100		376,081
	1,338,320		511,969

Sequence (b)

6 •	-80	41 •	31
	92,002		35,195
	14,300,802		5,470,715

Sequence (c):

6 •	248	41 •	95
	29,850		11,419
	44,070,130		16,861,531

Sequence (d)

6 •	2,052	41 •	785
	3,610		1,381
	364,459,330		139,422,469

Observe:
$$80 - 2 = 13 \cdot 6$$
$$80 + 2 = 2 \cdot 41$$
$$248 - 80 = 28 \cdot 6$$
$$248 + 80 = 8 \cdot 41$$
$$\ldots$$

4. Some construction rules for x_2, y_2, based on C:

4.1 $C = 1$: $x_2 = x_1(4B \cdot y_1^2 - 1)$ $\quad$ $y_2 = y_1(4A \cdot x_1^2 + 1)$

Example: $A = 23$, $B = 26$, $C = 1$

23 •	185 (x_1)	x_n, y_n = see 3.7	26 •	174 (y_1)
	582,510,055 (x_2)			587,873,974 (y_2)

4.2 $C = 2$: $x_2 = x_1(2B \cdot y_1^2 - 1)$ $\quad$ $y_2 = y_1(2A \cdot x_1^2 + 1)$

Example: $A = 33$, $B = 107$, $C = 2$

33 •	9 (x_1)	x_n, y_n = see 3.7	107 •	5 (y_1)
	48,141 (x_2)			26,735 (y_2)

4.3 $C = 4$; A, B, x, y = odd

$x_2 = x_1[(B \cdot y_1^2 - 1)(B \cdot y_1^2 - 2) - 1]$ $\quad$ $y_2 = y_1[(B \cdot y_1^2 - 2)(B \cdot y_1^2 - 3) - 1]$

Example: $A = 11$, $B = 47$, $C = 4$

11 •	31 (x_1)	47 •	15 (y_1)
	3,465,765,931 (x_2)		1,676,666,325 (y_2)

5. Apart from other formulas for x_2, y_2, based on other values of C, there exist those construction rules for groups (e.g., Problem H-350, and the problem based on F/L numbers I submitted in July 1982). However, this would be a field with no end, thus Problem H-354 has no general solution.

◆◇◆◇◆