A MODIFIED TRIBONACCI SEQUENCE

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(Submitted November 1982)

1. INTRODUCTION

The Tribonacci sequence [1] is generated by the recurrence relation

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \tag{1}$$
$$\text{with } U_0 = 0, \text{ and } U_1 = U_2 = 1.$$

Part of the charm of the original Fibonacci sequence $\{F_n\}$ is the ease with which new relations can be found, and a wealth of applications. However, (1) is rather unweildy and does not yield relations too readily. This article suggests a modification so that a development analogous to the Fibonacci sequence can be made. In addition, higher-order sequences can be constructed.

2. RECURRENCE RELATIONS FOR THE MODIFIED TRIBONACCI SEQUENCE

Consider $\{T_n\}$ generated by the recurrence relation

$$T_{2n} = T_{2n-1} + T_{2n-3}, \tag{2a}$$
$$T_{2n+1} = T_{2n-1} + T_{2n-2}, \tag{2b}$$
$$\text{where } n > 2, \text{ and } T_1 = T_2 = T_3 = 1.$$

The numerical sequence that emerges using (2a) and (2b) is:

1, 1, 1, 2, 2, 3, 4, 6, 7, 11, 13, 20, 24, 37, 44, 68, 81, ...

Note that $\{T_n\}$ resembles $\{F_n\}$ in its mode of definition.

However, successively odd and even terms are defined separately—note also that each odd term is the sum of the three previous odd terms, and, similarly, for the even terms. In this latter respect, the sequence resembles Tribonacci.

3. SOME PROPERTIES OF (T_n)

We can now go on to develop properties of $\{T_n\}$, some of which are analogous in form to $\{F_n\}$. These are presented without proof, as they are all elementary; no claim to completenes of the list is made.

$$T_{2n+5} = T_{2n+3} + T_{2n+1} + T_{2n-1}, \ n \geqslant 2; \tag{3}$$
$$T_{2n+6} = T_{2n+4} + T_{2n+2} + T_{2n}, \ n \geqslant 2; \tag{4}$$
$$T_2 + T_4 + \cdots + T_{2n} = T_{2n+3} - 1, \ n \geqslant 2; \tag{5}$$
$$T_1 + T_3 + \cdots + T_{2n-1} = (T_{2n} + T_{2n+2} - 1)/2, \ n \geqslant 2; \tag{6}$$
$$T_{2n+1}^2 - T_{2n-1}^2 = T_{2n+2} \cdot T_{2n-2}, \ n \geqslant 2; \tag{7}$$
$$T_2T_6 + T_4T_8 + \cdots + T_{2n-2} \cdot T_{2n+2} = T_{2n+1}^2 - 1, \ n \geqslant 2; \tag{8}$$
$$T_1T_3 + T_3T_5 + T_5T_7 + \cdots + T_{2n+1} \cdot T_{2n+3} = (T_{2n+4}^2 + T_{2n+2}^2 - 1)/4, \ n \geqslant 2; \tag{9}$$

$$T^2_{2n+2} + T^2_{2n-2} = 2(T^2_{2n+1} + T^2_{2n-1}),\ n \geqslant 2. \tag{10}$$

4. A GENERATING FUNCTION FOR $\{T_n\}$

A generating function corresponding to the development in [2] is now presented. We first consider the odd and even series separately:

$$F_e(x) = 1 + T_2x^2 + T_4x^4 + T_6x^6 + T_8x^8 \ldots \text{ when the subscript is even} \tag{11}$$

and

$$F_o(x) = T_1x + T_3x^3 + T_5x^5 + T_7x^7 + T_9x^9 \ldots \text{ when the subscript is odd.}$$

Therefore,

$$(1 - x^2 - x^4 - x^6) \cdot F_e(x) = 1 - x^6, \text{ by (4)} \tag{12}$$

or

$$F_e(x) = (1 - x^6)/(1 - x^2 - x^4 - x^6) \text{ when the subscript is even.}$$

Similarly, we have

$F_o(x) = x/(1 - x^2 - x^4 - x^6)$, by (3), when the subscript is odd. Hence,

$$F(x) = (1 + x - x^6)/(1 - x^2 - x^4 - x^6) \tag{13}$$

is the required generating function.

5. AN ALTERNATIVE PRESENTATION

Consider the original Fibonacci sequence $\{F_n\}$, with

$$F_0 = 0 \quad \text{and} \quad F_1 = 1,$$

then

$$F_{n+2} = F_{n+1} + F_n,\ n \geqslant 0. \tag{14}$$

It is well known that if

$$x^2 = 1 + x, \tag{15}$$

then

$$x^{n+1} = F_{n-1} + F_nx. \tag{16}$$

We see that the Fibonacci sequence is generated in this way. Similarly, we can generate $\{T_n\}$ by considering

$$x^3 = T_1 + T_2x + T_3x = 1 + x + x^2. \tag{17}$$

This gives

$$x^4 = T_3 + T_4x + T_5x^2 \tag{18}$$

$$x^5 = T_5 + T_6x + T_7x^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

leading to

$$x^{n+3} = T_{2n+1} + T_{2n+2}x + T_{2n+3}x^2,\ n \geqslant 2. \tag{19}$$

6. GENERALIZATIONS

By considering the method of Section 5 applied to

$$x^4 = 1 + x + x^2 + x^3, \tag{20}$$

we can construct the sequence $\{Q\ \}$, defined by

$$Q_1 = Q_2 = Q_3 = Q_4 = 1, \tag{21}$$

and (for $n \geqslant 1$),

$$Q_{3n+2} = Q_{3n+1} + Q_{3n-2},$$
$$Q_{3n+3} = Q_{3n+1} + Q_{3n-1}, \quad (22)$$
$$Q_{3n+4} = Q_{3n+1} + Q_{3n}.$$

leading to

$$X^{n+4} = Q_{3n+1} + Q_{3n+2}x + Q_{3n+3}x^2 + Q_{3n+4}x^3. \quad (23)$$

This sequence has the form

$$1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 6, 7, 8, 12, 14, 15, 23, 27, \ldots \quad (24)$$

We note that three Fibonacci-like recurrence relations are interwoven, and the feature

$$Q_{3n} = Q_{3n-3} + Q_{3n-6} + Q_{3n-9} + Q_{3n-12}, \; n \geqslant 4, \quad (25)$$

is retained. Further properties of this sequence can then be considered, as well as higher-order sequences.

REFERENCES

1. Mark Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* 1, no. 3 (1963):71-74.
2. W. R. Spickerman. "Binet's Formula for the Tribonacci Sequence." *The Fibonacci Quarterly* 20, no. 2 (1982):118-20.

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