AN APPLICATION OF THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

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1. Put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}), \\ 0 & (x = \text{integer}). \end{cases} \tag{1.1}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right). \tag{1.2}$$

It is well known that $s(h, k)$ satisfies the reciprocity theorem

$$s(h, k) = s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right), \tag{1.3}$$

where $(h, k) = 1$. For references, see [1, Ch. 2].

In this note, we shall show that (1.3) implies the following result.

Theorem 1

Let h, h', k, k' denote positive integers. (a) The system

$$\begin{cases} hh' \equiv 1 \pmod{k}, & hh' \equiv 1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'} \end{cases} \tag{1.4}$$

has no solutions with $h \neq h'$, $k \neq k'$. (b) The solutions of

$$\begin{cases} hh' \equiv -1 \pmod{k}, & hh' \equiv -1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'} \end{cases} \tag{1.5}$$

with $k \neq k'$ satisfy

$$kk' - hh' = 1, \tag{1.6}$$

and conversely.

The auxiliary inequalities in hypotheses (a) and (b) cannot be dispensed with. Thus, for example, (1.4) is satisfied by

$$(h, h', k, k') = (2, 3, 5, 5) \text{ and } (2, 4, 7, 7);$$

(1.5) is satisfied by

$$(h, h', k, k') = (3, 5, 4, 4) \text{ and } (2, 3, 7, 7).$$

Note that (3, 5, 4, 4) satisfies (1.6), but (2, 3, 7, 7) does not.

The congruences (1.4) and (1.5) suggest that it may be of interest to consider the following, more general, situation.

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \beta \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \delta \pmod{h'}, \end{cases} \tag{1.7}$$

where each of α, β, γ, and δ is equal to ± 1. We find that the method used in proving Theorem 1 applies, provided $\alpha\delta = \beta\gamma$ and $\alpha = \beta$. Thus, there are just four cases to consider. The cases $\alpha = \gamma = 1$ and $\alpha = -1$, $\gamma = 1$ are covered by Theorem 1. The case $\alpha = 1$, $\gamma = -1$ is essentially the same as $\alpha = -1$, $\gamma = 1$. The one remaining case is covered by the following:

Theorem 2

The system of congruences

$$\begin{cases} hh' \equiv -1 \pmod{k}, & hh' \equiv -1 \pmod{k'} \\ kk' \equiv -1 \pmod{h}, & kk' \equiv -1 \pmod{h'} \end{cases} \tag{1.8}$$

has no solutions in positive integers h, h', k, k'.

Note that it is now not necessary to assume either $k' \neq k$ or $h' \neq h$.

2. It follows from (1.1) that

$$((-x)) = -((x)). \tag{2.1}$$

Thus,

$$s(-h, k) = -s(h, k). \tag{2.2}$$

In the next place, if $hh' \equiv 1 \pmod{k}$, then, by (1.2),

$$s(h', k) = \sum_{r \pmod{k}} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{h'r}{k}\right)\right) = \sum_{t \pmod{k}} \left(\left(\frac{ht}{k}\right)\right)\left(\left(\frac{t}{k}\right)\right),$$

on replacing r by ht and using the periodicity of $((x))$. Hence,

$$s(h', k) = s(h, k) \qquad [hh' \equiv 1 \pmod{k}]. \tag{2.3}$$

Similarly,

$$s(h', k) = -s(h, k) \qquad [hh' \equiv -1 \pmod{k}]. \tag{2.4}$$

Now let h, h', k, k' be positive integers that satisfy the system of congruences

$$\begin{cases} hh' \equiv 1 \pmod{k}, & hh' \equiv 1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'}. \end{cases} \tag{2.5}$$

Thus,

$$(h, k) = (h, k') = (h', k) = (h', k') = 1.$$

Therefore, we may apply the reciprocity theorem (1.3) to get the following set of equations:

$$\begin{cases} s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ s(h', k) + s(k, h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h, k') + s(k', h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ s(h', k') + s(k', h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{2.6}$$

In view of (2.3), we have

$$\begin{cases} s(h,\ k) + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ s(h,\ k) + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h,\ k') + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ s(h,\ k') + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{2.7}$$

Multiplying the first and fourth equations in (2.6) by +1, the second and third by -1, and adding the resulting equations, we get

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) - \left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) + \left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0,$$

or better,

$$h'k'(h^2 + 1 + k^2) - hk'(h'^2 + 1 + k^2) - h'k(h^2 + 1 + k'^2) + hk(h'^2 + 1 + k'^2) = 0.$$

A little manipulation yields

$$(h' - h)(k' - k)(1 - hh' - kk') = 0. \tag{2.8}$$

Now, assuming that $h' \neq h$ and $k' \neq k$, (2.8) reduces to

$$hh' + kk' = 1. \tag{2.9}$$

Since (2.9) obviously has no solutions in positive integers h, h', k, k', we have proved the first half of Theorem 1.

3. To prove the second part of the theorem, let h, h', k, k' be positive integers that satisfy the congruences

$$\begin{cases} hh' \equiv -1 \pmod{k}, \quad hh' \equiv -1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, \quad kk' \equiv 1 \pmod{h'}. \end{cases} \tag{3.1}$$

Then

$$(h,\ k) = (h,\ k') = (h',\ k) = (h',\ k') = 1,$$

and exactly as above, we get the set of equations (2.6). However, we now use both (2.3) and (2.4). Thus, in place of (2.7), we get

$$\begin{cases} s(h,\ k) + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ -s(h,\ k) + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h,\ k') + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ -s(h,\ k') + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{3.2}$$

Multiply the first and second equations by +1, the third and fourth by -1, and add:

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) + \left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) - \left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0,$$

or

$$h'k'(h^2 + 1 + k^2) + hh'(h'^2 + 1 + k^2) - h'k(h^2 + 1 + k'^2) - hk(h'^2 + 1 + k'^2) = 0.$$

This reduces to

$$(h' + h)(k' - k)(1 + hh' - kk') = 0.$$

Hence, assuming $k' \neq k$, we get

$$kk' - hh' = 1. \tag{3.3}$$

This completes the proof of the theorem.

4. We now consider the system of congruences

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \beta \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \delta \pmod{h'}, \end{cases} \tag{4.1}$$

where each of α, β, γ, δ is equal to ± 1. Then, in place of (2.7), we get

$$\begin{cases} s(h, k) + s(k, h) \\ \alpha s(h, k) + s(k, h') \\ s(h, k') + \gamma s(k, h) \\ \beta s(h, k') + \delta s(k, h'), \end{cases} \tag{4.2}$$

where, for brevity, we indicate only the left-hand sides.

Now, multiply the first equation in (4.2) by 1, the second by ξ, the third by η, the fourth by ζ. To eliminate the left-hand side, ξ, η, ζ must satisfy

$$1 + \alpha\xi = 0,\ 1 + \gamma\eta = 0,\ \xi + \delta\zeta = 0,\ \eta + \beta\zeta = 0.$$

This gives

$$\xi = -\alpha,\ \eta = -\gamma,\ \zeta = \alpha\delta,\ \zeta = \beta\gamma, \tag{4.3}$$

so that $\alpha\delta = \beta\gamma$, $\delta = \alpha\beta\gamma$. Hence, (4.3) becomes

$$\xi = -\alpha,\ \eta = -\gamma,\ \zeta = \beta\gamma. \tag{4.4}$$

It follows that (4.2) implies

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) - \alpha\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \gamma\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) + \beta\gamma\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0.$$

Simplifying, we get

$$(h'k' - \alpha hk' - \gamma h'k + \beta\gamma hk) - \alpha hh'(h'k' - \alpha hk' - \alpha\beta\gamma h'k + \alpha\gamma hk)$$
$$- \gamma kk'(h'k' - \beta hk' - \gamma h'k + \alpha\gamma hk) = 0.$$

If $\alpha = \beta$, this becomes

$$(h' - \alpha h)(k' - \gamma k)(1 - \alpha hh' - \gamma kk') = 0, \tag{4.5}$$

while (4.1) reduces to

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \alpha \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \gamma \pmod{h'}. \end{cases} \tag{4.6}$$

The cases $\alpha = \gamma = 1$ and $\alpha = -1$, $\gamma = 1$ are covered by Thoerem 1. The case $\alpha = 1$, $\gamma = -1$ is essentially the same as $\alpha = -1$, $\gamma = 1$. Thus, the only case to consider is $\alpha = \gamma = -1$. In this case (4.5) is

$$(h' + h)(k' + k)(1 + hh' + kk') = 0. \tag{4.7}$$

Clearly, (4.7) cannot be satisfied in positive integers. It is now not necessary to assume either $k' \neq k$ or $h' \neq h$.

This completes the proof of Theorem 2.

REFERENCE

1. H. Rademacher & E. Grosswald. *Dedekind Sums*. Washington, D.C.: The Mathematical Association of America, 1972.

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COAXAL CIRCLES ASSOCIATED WITH RECURRENCE-GENERATED SEQUENCES

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1. INTRODUCTION

Recently, some articles [1], [3], and [4] of a geometrical nature relating Fibonacci numbers to circles, with an extension to conics, have appeared in this journal. Here, we offer another geometrical connection between Fibonacci-type numbers and circles (though this material bears no relation to the other articles). In particular, it is shown how Fibonacci and Lucas numbers, and their generalization, are associated with sets of coaxal circles.

Define the recurrence-generated sequence $\{H_n\}$ for all values of n (integer) by

$$H_{n+2} = H_{n+1} + H_n,\ H_0 = 2b,\ H_1 = a + b, \tag{1.1}$$

where a and b are arbitrary, but may be thought of as integers.

Using [2], equation (δ), we have, *mutatis mutandis*, the explicit Binet form for this generalized sequence

$$H_n = \frac{(a + \sqrt{5}b)\alpha^n - (a - \sqrt{5}b)\beta^n}{\sqrt{5}}, \tag{1.2}$$

where $\alpha = (1 + \sqrt{5})/2$ (> 0), $\beta = (1 - \sqrt{5})/2$ (< 0) are the roots of $x^2 - x - 1 = 0$ (so that $\alpha\beta = -1$).

From (1.2) it follows that

$$H_n = aF_n + bL_n, \tag{1.3}$$

where

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \tag{1.4}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.5}$$

are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, occurring in (1.1), (1,2), and (1.3) when $a = 1$, $b = 0$ (for F_n) and $a = 0$, $b = 1$ (for L_n).

Observe from (1.4) and (1.5) that

$$\sqrt{5}F_n < L_n \text{ when } n \text{ is even}, \tag{1.6}$$

while

$$\sqrt{5}F_n > L_n \text{ when } n \text{ is odd}. \tag{1.7}$$

2. COAXAL CIRCLES FOR $\{H_n\}$

Consider the point with Cartesian coordinates $(x, 0)$ where x is given by

$$x = [(a + \sqrt{5}b)\alpha^{2n} + (a - \sqrt{5}b)\cos(n - 1)\pi]/\sqrt{5}\alpha^n \tag{2.1}$$