SUMS OF FIBONACCI NUMBERS BY MATRIX METHODS

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Recently, Kalman [2] derives a number of closed-form formulas for the generalized Fibonacci sequence by matrix methods. In this note, we extend the matrix representation and show that the sums of the generalized Fibonacci numbers could be derived directly using this representation.

Define k sequences of the generalized order-k Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} c_j g_{n-j}^i, \text{ for } n > 0 \text{ and } 1 \leqslant i \leqslant k, \tag{1}$$

with boundary conditions

$$g_n^i = \begin{cases} 1, & i = 1 - n, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for } 1 - k \leqslant n \leqslant 0,$$

where c_j, $1 \leqslant j \leqslant k$, are constant coefficients, and g_n^i is the n^{th} term of the i^{th} sequence. When $k = 2$, the generalized order-k Fibonacci sequence is reduced to the conventional Fibonacci sequence.

Following the approach taken by Kalman [2], we define a $k \times k$ square matrix A as follows:

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_{k-1} & c_k \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then, by a property of matrix multiplication, we have

$$\left[g_{n+1}^i \; g_n^i \; \dots \; g_{n-k+2}^i\right]^T = A\left[g_n^i \; g_{n-1}^i \; \dots \; g_{n-k+1}^i\right]^T. \tag{2}$$

To deal with the k sequences of the generalized order-k Fibonacci series simultaneously, we define a $k \times k$ square matrix G_n as follows:

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \dots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \dots & g_{n-1}^k \\ \vdots & \vdots & & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \dots & g_{n-k+1}^k \end{bmatrix}.$$

Generalizing Eq. (2), we derive

$$G_{n+1} = AG_n. \tag{3}$$

Then, by an inductive argument, we may rewrite it as

$$G_{n+1} = A^n G_1. \tag{4}$$

Now, it can be readily seen that, by Definition (1), $G_1 = A$; therefore, $G_n = A^n$. We may thus rewrite Eqs. (3) and (4) as shown:

$$G_{n+1} = G_1 G_n = G_n G_1. \tag{5}$$

In other words, G_1 is commutative under matrix multiplication. Hence, we have:

$$\begin{aligned} g_{n+1}^i &= c_i\, g_n^1 + g_n^{i+1}, \text{ for } 1 \leqslant i \leqslant k-1, \\ g_{n+1}^k &= c_k\, g_n^1. \end{aligned} \tag{6}$$

More generally, we may write Eq. (5) as $G_{r+c} = G_r G_c$. Consequently, an element of G_{r+c} is the product of a row of G_r and a column of G_c:

$$g_{r+c}^i = \sum_{j=1}^{k} g_r^j g_{c-j+1}^i.$$

In particular, when $r = c = n$, we have $G_{2n} = G_n^2$; this provides us with a means of evaluating G_n in an order of $\log_2 n$ steps.

To calculate the sums S_n, $n \geqslant 0$, of the generalized order-k Fibonacci numbers, defined by

$$S_n = \sum_{i=0}^{n} g_i^1, \tag{7}$$

let B be a $(k+1) \times (k+1)$ square matrix, such that

$$B = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ 1 & & \\ 0 & & A \\ \vdots & & \\ 0 & & \end{bmatrix}.$$

Further, let E_n also be a $(k+1) \times (k+1)$ square matrix, such that

$$E_n = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ S_{n-1} & & \\ S_{n-2} & & G_n \\ \vdots & & \\ S_{n-k} & & \end{bmatrix}.$$

Then, by Eq. (6) and

$$S_{n+1} = g_{n+1}^1 + S_n, \tag{8}$$

we derive a recurrence equation

$$E_{n+1} = E_n B. \tag{9}$$

Inductively, we also have

$$E_{n+1} = E_1 B^n. \tag{10}$$

Since $S_{-i} = 0$, $1 \leqslant i \leqslant k$, we thus infer $E_1 = B$, and in general, $E_n = B^n$. So, from Eqs. (9) and (10), we reach the following equation:

$$E_{n+1} = E_1 E_n = E_n E_1, \tag{11}$$

which shows that E_1 is commutative as well under matrix multiplication. By an application of Eq. (11), the sums of the generalized order-k Fibonacci numbers satisfy the following recurrence relation:

$$S_n = 1 + \sum_{i=1}^{k} S_{n-i}. \tag{12}$$

Substituting $S_n = g_n^1 + S_{n-1}$, an instance of Eq. (8), into Eq. (12), we may express g_n^1 in terms of the sums of the generalized order-k Fibonacci numbers:

$$g_n^1 = 1 + \sum_{i=2}^{k} S_{n-i}. \tag{13}$$

When $k = 2$, this equation is reduced to

$$g_n^1 = 1 + S_{n-2}.$$

If $c_1 = c_2 = 1$, we derive the well-known result [1]:

$$F_n = 1 + \sum_{i=0}^{n-2} F_i, \tag{14}$$

where F_n is the n^{th} term of the standard Fibonacci sequence. Equation (14) is also apparent from the Fibonacci number system viewpoint. Let

$$W = \{b_m \ \dots \ b_2 b_1 b_0\}$$

be a bit pattern, where b_i is either 0 or 1 associating with a weight F_i. Thus, by an analogy of the binary number system, any natural number N may be defined as

$$N = \sum_{i=0}^{m} b_i F_i,$$

where m is sufficiently large. Since

$$S_n = \sum_{i=0}^{n} F_i,$$

the bit pattern of S_n consists of $(n + 1)$ 1's, that is, $\{1\}_0^n$. By Zeckendorf's theorem [1, p. 74], the bit pattern can be normalized to a pattern made up of 1's at b_{n+2-i}, where i is odd, and 0's at other positions. If a 1 is added to this pattern and, after the same normalization, the whole bit pattern consists of a 1 at b_{n+2} and 0's at other positions; the value is clearly equal to F_{n+2}. By induction, Eq. (14) holds.

Further, Eq. (11) could be generalized to $E_{r+c} = E_r E_c$. If $r = c = n$, we have $E_{2n} = E_n^2$. Thus, E_n may be evaluated in an order of $\log_2 n$ steps, too.

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REFERENCES

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