REFERENCE

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COAXAL CIRCLES ASSOCIATED WITH RECURRENCE-GENERATED SEQUENCES

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1. INTRODUCTION

Recently, some articles [1], [3], and [4] of a geometrical nature relating Fibonacci numbers to circles, with an extension to conics, have appeared in this journal. Here, we offer another geometrical connection between Fibonacci-type numbers and circles (though this material bears no relation to the other articles). In particular, it is shown how Fibonacci and Lucas numbers, and their generalization, are associated with sets of coaxal circles.

Define the recurrence-generated sequence $\{H_n\}$ for all values of n (integer) by

$$H_{n+2} = H_{n+1} + H_n,\ H_0 = 2b,\ H_1 = a + b, \tag{1.1}$$

where a and b are arbitrary, but may be thought of as integers.

Using [2], equation (δ), we have, *mutatis mutandis*, the explicit Binet form for this generalized sequence

$$H_n = \frac{(a + \sqrt{5}b)\alpha^n - (a - \sqrt{5}b)\beta^n}{\sqrt{5}}, \tag{1.2}$$

where $\alpha = (1 + \sqrt{5})/2$ (> 0), $\beta = (1 - \sqrt{5})/2$ (< 0) are the roots of $x^2 - x - 1 = 0$ (so that $\alpha\beta = -1$).

From (1.2) it follows that

$$H_n = aF_n + bL_n, \tag{1.3}$$

where

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \tag{1.4}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.5}$$

are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, occurring in (1.1), (1,2), and (1.3) when $a = 1$, $b = 0$ (for F_n) and $a = 0$, $b = 1$ (for L_n).

Observe from (1.4) and (1.5) that

$$\sqrt{5}F_n < L_n \text{ when } n \text{ is even}, \tag{1.6}$$

while

$$\sqrt{5}F_n > L_n \text{ when } n \text{ is odd}. \tag{1.7}$$

2. COAXAL CIRCLES FOR $\{H_n\}$

Consider the point with Cartesian coordinates $(x, 0)$ where x is given by

$$x = [(a + \sqrt{5}b)\alpha^{2n} + (a - \sqrt{5}b)\cos(n - 1)\pi]/\sqrt{5}\alpha^n \tag{2.1}$$

[the x-value being another form of H_n in (1.2)].

Elementary calculations show that the circle, denoted by CH_n, with

$$\text{center: } \left(\left(\frac{a + \sqrt{5}b}{\sqrt{5}}\right)\alpha^n,\ 0\right) \equiv (\bar{x}(H_n),\ \bar{y}(H_n)) \tag{2.2}$$

and

$$\text{radius: } r(H_n) = \left|\frac{a - \sqrt{5}b}{\sqrt{5}\alpha^n}\right| \tag{2.3}$$

is

$$\left(x - \left(\frac{a + \sqrt{5}b}{\sqrt{5}}\right)\alpha^n\right)^2 + y^2 = \left(\frac{a - \sqrt{5}b}{\sqrt{5}\alpha^n}\right)^2. \tag{2.4}$$

Clearly,

$$\bar{x}(H_n)/\bar{x}(H_{n-1}) = \alpha \tag{2.5}$$

and

$$r(H_n)/r(H_{n-1}) = 1/\alpha, \tag{2.6}$$

so that the sets $\{\bar{x}(H_n)\}$ and $\{r(H_n)\}$ form geometrical progressions.

The circles CH_n cut the x-axis where

$$\begin{aligned} x(H_n) &= (a + \sqrt{5}b)\alpha^n/\sqrt{5} \pm (a - \sqrt{5}b)/\sqrt{5}\alpha^n \\ &= a(\alpha^n \pm (-1)^n\beta^n)/\sqrt{5} + b(\alpha^n \mp (-1)^n\beta^n), \text{ since } \alpha\beta = -1. \end{aligned}$$

That is,

$$\begin{aligned} x(H_n) &= aF_n + bL_n \quad \text{or} \quad aL_n/\sqrt{5} + \sqrt{5}bF_n \\ &= H_n \quad \text{or} \quad aL_n/\sqrt{5} + \sqrt{5}bF_n \text{ [by (1.3)].} \end{aligned} \tag{2.7}$$

The coordinates $x = \bar{x}(H_n)$, $y = r(H_n)$ of the highest point on CH_n lie on the upper branch of the rectangular hyperbola

$$xy = \left(\frac{a + \sqrt{5}b}{5}\right)|a - \sqrt{5}b| \tag{2.8}$$

on making use of (2.2) and (2.3).

Of the other three points of intersection of the circle (2.4) and the rectangular hyperbola (2.8), only one is real, given by the real root of the cubic equation $x^3 - \alpha^n x^2 - \alpha^{-2n}x - \alpha^{-n} = 0$, e.g., in the case of $\{L_n\}$. No obvious geometry follows from the set of these real points [though one might hope that their locus would be a simple curve (another rectangular hyperbola?)].

Similar results apply to the case of the lowest point.

3. COAXAL CIRCLES FOR $\{F_n\}$ AND $\{L_n\}$

Parallel details for the special cases $\{F_n\}$ and $\{L_n\}$ of $\{H_n\}$ arising when $a = 1$, $b = 0$, and $a = 0$, $b = 1$, respectively, can be tabulated, as in the following table, after making appropriate notational adjustments to the results (2.1)-(2.8) in the previous section.

Interesting features of the table appear in (3.7):

(i) the (integer) Fibonacci numbers and the irrational numbers of the Lucas-related sequence $\{L_n\}/\sqrt{5}$ are represented on the x-axis as the points of intersection of the axis and the set of coaxal circles CF_n, and

(ii) the (integer) Lucas numbers and the irrational numbers of the Fibonacci-related sequence $\sqrt{5}\{F_n\}$ are represented on the x-axis as the points of intersection of the axis and the set of coaxal circles CL_n.

If we define the orientation of a circle of the coaxal sets to be that in going (above the x-axis) from the Fibonacci value to the Lucas value in (3.7),

	$\{F_n\}$	$\{L_n\}$
(3.1)	$\begin{cases} x = (\alpha^{2n} + \cos(n-1)\pi)/\sqrt{5}\alpha^n \\ y = 0 \end{cases}$	$\begin{cases} x = (\alpha^{2n} - \cos(n-1)\pi)/\alpha^n \\ y = 0 \end{cases}$
(3.2)	$\bar{x}(F_n) = \alpha^n/\sqrt{5},\ y(F_n) = 0$	$\bar{x}(L_n) = \alpha^n,\ \bar{y}(L_n) = 0$
(3.3)	$r(F_n) = 1/\sqrt{5}\alpha^n$	$r(L_n) = 1/\alpha^n$
(3.4)	$CF_n:\ \left[x - \dfrac{\alpha^n}{\sqrt{5}}\right]^2 + y^2 = \dfrac{1}{5\alpha^{2n}}$	$CL_n:\ (x - \alpha^n)^2 + y^2 = 1/\alpha^{2n}$
(3.5)	$\bar{x}(F_n)/\bar{x}(F_{n-1}) = \alpha$	$\bar{x}(L_n)/\bar{x}(L_{n-1}) = \alpha$
(3.6)	$r(F_n)/r(F_{n-1}) = 1/\alpha$	$r(L_n)/r(L_{n-1}) = 1/\alpha$
(3.7)	$x(F_n) = F_n,\ L_n/\sqrt{5}$	$x(L_n) = L_n,\ \sqrt{5}F_n$
(3.8)	$xy = 1/5$	$xy = 1$

then (1.6) and (1.7) disclose that the orientation is reversed for alternate circles in both coaxal sets.

It is an instructive exercise to draw some of the circles CF_n and CL_n for small integral values of n (<0, $=0$, >0), but we omit the diagram here in order to conserve space.

4. CONCLUDING REMARKS

This article developed from a brief private communication from L. G. Wilson [5], to whom the author expresses his thanks. Wilson, however, was concerned only with the polar coordinate representation of the points given in Cartesian coordinates (x, y) by x as in (2.1), and $y = (a - \sqrt{5}b)\sin(n - 1)\pi/\sqrt{5}\alpha^n$ but with n not restricted to integral values. Our concentration on just two special points on each circle was stimulated by a desire to exhibit the circle generation of the members of $\{F_n\}$ and $\{L_n\}$.

The occurrence of $\alpha^n/\sqrt{5}$ and α^n reminds us that these, by (1.4) and (1.5), are the limiting values of F_n and L_n, respectively. Thus, if n is graphed against $y = \lim_{n\to\infty} F_n$ and $y = \lim_{n\to\infty} L_n$ in turn, we find that the points (F_n, y) and (L_n, y) lie remarkably close to the exponential curves $y = \alpha^n/\sqrt{5}$ and $y = \alpha^n$ even for small values of n.

It seems reasonable to expect an extension, albeit a slightly tedious one, to the more general sequence $\{W_n\}$ defined for all integral n by

$$W_{n+2} = pW_{n+1} - qW_n, \tag{4.1}$$

with specified values for W_0 and W_1. Possibly some worthwhile results for the special cases of the Pell sequences arising from (4.1) when $p = 2$, $q = 1$ might eventuate from this investigation.

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