A NOTE ON SOMER'S PAPER ON LINEAR RECURRENCES

ERNST S. SELMER
University of Bergen, 5014 Bergen, Norway
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In a recent paper [1], Somer uses the second-order linear recursion relation

$$s_{n+2} = as_{n+1} + bs_n,\ a,\ b \in \mathbf{Z}, \tag{1}$$

to generate higher-order linear recurrences. The purpose of this note is to extend Somer's results. In what follows, the notation in [1] is used without further comment.

We assume $\alpha\beta \neq 0$, α/β not a root of unity, and ask under what conditions the *rational* sequence

$$\{t_n\}_{n=0}^{\infty} = \{s_{nk}/s_n\}_{n=0}^{\infty} \tag{2}$$

satisfies a linear recursion relation of *minimal* order k.

Somer gives the solution $\{s_n\} = \{u_n\}$, where $u_0 = 0$, $u_1 = 1$. We can argue similarly for $\{s_n\} = \{v_n\}$, where $v_0 = 2$, $v_1 = a$, and $v_n = \alpha^n + \beta^n$, in the case when k is *odd*. Then

$$t_n = \frac{v_{nk}}{v_n} = \frac{\alpha^{nk} + \beta^{nk}}{\alpha^n + \beta^n} = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n}(-1)^i \beta^{in}$$

is a rational integer, and $\{t_n\}$ clearly satisfies the same k^{th}-order linear recursion relation as $\{w_n\} = \{u_{nk}/u_n\}$. The proof of the minimality runs as for $\{w_n\}$: In the *first* matrix factor of $D_k(w_n, 0)$, we just change the sign of every odd-numbered column.

The general solution $s_n \neq s_1 u_n$ of (1) may be written as

$$s_n = \frac{A\alpha^n + B\beta^n}{A + B},$$

if we "normalize" to $s_0 = 1$. The above result for $\{v_n\}$ then follows from the fact that $-B/A = -1$ is a primitive square root of unity. In general, put $-B/A = \rho$, where ρ is a primitive m^{th} root of unity, and assume that

$$k \equiv 1 \pmod{m}.$$

Then

$$t_n = \frac{s_{nk}}{s_n} = \frac{\alpha^{nk} - \rho\beta^{nk}}{\alpha^n - \rho\beta^n} = \frac{\alpha^{nk} - (\rho\beta^n)^k}{\alpha^n - \rho\beta^n} = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n}\rho^i\beta^{in}.$$

The question of minimality is settled as above: To obtain $D_k(t_n, 0)$, we multiply the successive columns of the first matrix factor of $D_k(w_n, 0)$ by 1, ρ, ρ^2, ..., ρ^{k-1}, respectively.

For $m > 2$, however, the *rationality* of $\{s_n\}$ imposes severe conditions. In particular,

$$s_1 = \frac{A\alpha + B\beta}{A + B} = \frac{\alpha - \rho\beta}{1 - \rho}$$

should be rational, showing that $\rho = \sqrt[m]{1}$ must be a quadratic irrationality, so $m = 3, 4$, or 6. But even in these cases, we get conditions on the coefficients a and b.

We illustrate the method in the case $m = 4$, $\rho = \pm i$. With

$$\alpha = \frac{a + \sqrt{D}}{2}, \quad \beta = \frac{a - \sqrt{D}}{2}, \quad D = a^2 + 4b,$$

this gives $s_1 = (a \pm i\sqrt{D})/2$, which is rational if and only if $D = -c^2$, $c \in \mathbf{Z}$. Then

$$s_1 = \frac{a + c}{2} = \frac{v_1 + cu_1}{2} \quad \left(\text{and} \quad s_0 = \frac{v_0 + cu_0}{2} = 1\right).$$

To get b integral, both a and c must be even. To get $\alpha\beta \neq 0$ and α/β not a root of unity, we must have $ac \neq 0$ and $a \neq \pm c$. Consequently, we have shown that if

$$c \in \mathbf{Z}, \; b = -\frac{a^2 + c^2}{4}, \; 2|a, \; 2|c, \; ac \neq 0, \; a \neq \pm c,$$

then the integral sequence

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + cu_n}{2}\right\}_{n=0}^{\infty}$$

has the property (2) when $k \equiv 1 \pmod 4$.

We only state the corresponding results for $m = 3$ and $m = 6$. Let

$$c \in \mathbf{Z}, \; b = -\frac{a^2 + 3c^2}{4},$$

a and c be of the same parity, $ac \neq 0$, $a \neq \pm c$, $\pm 3c$. Then the following integral sequences have the property (2):

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + cu_n}{2}\right\}_{n=0}^{\infty} \quad \text{if } k \equiv 1 \pmod 3,$$

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + 3cu_n}{2}\right\}_{n=0}^{\infty} \quad \text{if } k \equiv 1 \pmod 6.$$

REFERENCE

1. L. Somer. "The Generation of Higher-Order Linear Recurrences from Second-Order Linear Recurrences." *The Fibonacci Quarterly* 22, no. 2 (1984):98-100.

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