SOME ASYMPTOTIC PROPERTIES OF GENERALIZED FIBONACCI NUMBERS

A. G. SHANNON
The New South Wales Institute of Technology, N.S.W. 2007, Australia
(Submitted November 1982)

1. INTRODUCTION

Horadam [1] has generalized two theorems of Subba Rao [3] which deal with some asymptotic properties of Fibonacci numbers. Horadam defined a sequence

$$\{w_n^{(2)}\} = \{w_n(w_0,\ w_1;\ P_{21},\ P_{22})\}$$

which satisfies the second-order recurrence relation

$$w_{n+2} = P_{21}w_{n+1} - P_{22}w_n, \text{ with } w_n = A_{21}\alpha_{21}^n + A_{22}\alpha_{22}^n,$$

where α_{21}, α_{22} are the roots of $x^2 - P_{21}x + P_{22} = 0$. We shall let

$$d = \alpha_{22} - \alpha_{21}.$$

Horadam established two theorems for $\{w_n\}$:

I. The number of terms of $\{w_n\}$ not exceeding N is asymptotic to

$$\underline{\log}(Nd/(P_{22}w_0 - \alpha_{21}w_1)).$$

II. The range, within which the rank n of w_n lies, is given by

$$\underline{\log}\ w_n + \underline{\log}(X - d)/x < n + 1 < \underline{\log}\ w_n + \underline{\log}(Y - d)/x,$$

where

$$X = y/(w_{-1} + 2x),\ Y = y/(w_{-1} - 2x),$$

$$x = w_0 - \alpha_{22}w_{-1},\ y = w_0 - \alpha_{21}w_{-1},$$

and in which $\underline{\log}$ stands for logarithm to the base α_{r1}; $r = 2$ in this case.

These were generalizations of two theorems which Subba Rao had proved for

$$\{f_n\} : f_n = w_n(1,\ 1;\ 1,\ -1),$$

the ordinary Fibonacci numbers.

It is proposed here to explore generalizations of the Horadam-Subba Rao theorems to sequences, the elements of which satisfy linear recurrence relations of artitrary order. To this end, we define $\{w_n^{(r)}\}$:

$$w_{n+r}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}P_{rj}\,w_{n+r-j}^{(r)},\ n \geqslant 0, \tag{1.1}$$

with suitable initial values $w_n^{(r)}$, $n = 0, 1, \ldots, r - 1$, and where the P_{rj} are arbitrary integers. Thus, $\{w_n^{(2)}\}$ represents Horadam's generalized sequence of integers.

We can suppose then that

$$w_n^{(r)} = \sum_{j=1}^{r} A_{rj}\alpha_{rj}^n, \tag{1.2}$$

in which the A_{rj} depend on the initial values of $\{w_n^{(r)}\}$, and the α_{rj} are the roots (assumed distinct) of

$$x^r - \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, x^{r-j} = 0. \tag{1.3}$$

In fact, $A_{ri} = d_i/d$, where

$$d = \prod_{\substack{i,j=1\\ i>j}}^{r} (\alpha_{ri} - \alpha_{rj})$$

is the Vandermonde of the roots α_{ri}, and d_i is obtained from d on replacement of its i^{th} column by the r initial terms of $\{w_n^{(r)}\}$ (Jarden [2]).

2. ASYMPTOTIC BEHAVIOR

Where convenient in this section, we follow the reasoning of Horadam or of Subba Rao.

Theorem A

The number of terms of $\{w_n^{(r)}\}$ not exceeding N is asymptotic to

$$\underline{\log}(N/A_{r1}\alpha_{r1}).$$

Proof: Suppose $w_n^{(r)} \leqslant N < w_{n+1}^{(r)}$. $\quad$ (2.1)

Then the left-hand side yields

$$w_n^{(r)} = \sum_{j=1}^{r} A_{rj}\,\alpha_{rj}^{n} \leqslant N.$$

Suppose further that $|\alpha_{r1}| > |\alpha_{r2}| > \cdots > |\alpha_{rr}| > 0$, so that for $m > 1$,

$$(\alpha_{rm}/\alpha_{r1})^n \to 0 \text{ as } n \to \infty,$$

and

$$A_{r1} \leqslant N/\alpha_{r1}^{n}.$$

Hence,

$$n \log \alpha_{r1} + \log A_{r1} \leqslant \log N$$

and

$$n \leqslant \underline{\log}(N/A_{r1}). \tag{2.2}$$

The right-hand side of the first inequality (2.1) yields

$$N < \sum_{j=1}^{r} A_{rj}\,\alpha_{rj}^{n+1};$$

whence

$$n + 1 > \underline{\log}(N/A_{r1}). \tag{2.3}$$

Thus, from inequalities (2.2) and (2.3):

$$n - 1 \leqslant \underline{\log}(N/A_{r1}) - 1 < n,$$

or

$$n \sim \underline{\log}(N/A_{r1}\alpha_{r1}) \text{ as required.}$$

This is a generalization of Theorem I of Horadam, because when $r = 2$ and $w_0^{(2)} = a$, $w_1^{(2)} = b$,

$$A_{21}\alpha_{21} = d_1\alpha_{21}/d = (a\alpha_{22} - b)\alpha_{21}/d = (aP_{22} - \alpha_{21}b)/d),$$

which agrees with Horadam.

3. RANK

To obtain a partial generalization of Theorem II of Horadam and the corresponding proposition of Subba Rao, we first define $\{U_n^{(r)}\}$, a fundamental sequence of order r; we illustrate its fundamental nature by showing that any linear recursive sequence of order r can be expressed in terms of $\{U_n^{(r)}\}$.

We define $\{U_n^{(r)}\}$ by means of

$$\begin{aligned} \alpha_{rj}^n &= \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k/\omega^{kj}, \; n \geqslant 0, \\ &= 0, \; n < 0. \end{aligned} \tag{3.1}$$

where $D = \sum_{j=1}^{r} \omega^j \alpha_{rj}$, in which $\omega = \exp(2\pi i/r)$ and α_{rj} satisfies (1.3).

It follows that

$$U_n^{(r)} = D^{-1} \sum_{j=1}^{r} \omega^j \alpha_{rj}^n, \; n \geqslant 0, \tag{3.2}$$

Proof: $\displaystyle\sum_{j=1}^{r} \omega^j \alpha_{rj}^n = \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k \sum_{j=1}^{r} \omega^{(1-k)j} = \frac{1}{r} U_n^{(r)} Dr,$

which gives the result, since

$$\frac{1}{r} \sum_{j=1}^{r} \omega^{ij} = \delta_{i0}, \text{ the Kronecker delta.}$$

For example, when $r = 2$, $\omega = -1$, and we get, from (3.2), that

$$\begin{aligned} U_0^{(2)} &= D^{-1}(-1 + 1) = 0, \\ U_1^{(2)} &= D^{-1}(-\alpha_{21} + \alpha_{22}) = 1, \\ U_2^{(2)} &= D^{-1}(-\alpha_{21} + \alpha_{22}) = P_{21}, \end{aligned}$$

so that $U_n^{(2)} = u_{n-1}$ defined in (1.8) of Horadam, because, for $n > 1$, $U_n^{(r)}$ satisfies the recurrence relation (1.1).

Proof: The right-hand side of this recurrence relation is

$$\begin{aligned} \sum_{j=1}^{r} (-1)^{j+1} P_{rj} U_{n-j}^{(r)} &= \sum_{k=1}^{r} \sum_{j=1}^{r} (-1)^{j+1} P_{rj} D^{-1} \alpha_{rk}^{n-j} \omega^k \\ &= \frac{1}{D} \sum_{k=1}^{r} \left(\sum_{j=1}^{r} (-1)^{j+1} P_{rj} \alpha_{rk}^{r-j} \right) \alpha_{rk}^{n-r} \omega^k \\ &= \frac{1}{D} \sum_{k=1}^{r} \alpha_{rk}^r \alpha_{rk}^{n-r} \omega^k \\ &= U_n^{(r)} \end{aligned}$$

Our next result is quite important in that it justifies our finding the rank of $U_n^{(r)}$ instead of that of $w_n^{(r)}$ because every $w_n^{(r)}$ can be expressed in terms of the fundamental $U_n^{(r)}$.

To prove this, we look first at the set

$$P = P(P_{r1}, P_{r2}, \ldots, P_{rr})$$

of all sequences of order r which satisfy

$$\sum_{j=0}^{r}(-1)^j P_{rj}\, w_{n-j}^{(r)} = 0,\ P_{r0} = -1.$$

P is closed with respect to "addition" and "scalar multiplication" and

$$\{U_{n-k}^{(r)}\} \in P,\ k < n+1,$$

so we may seek to express the elements of P as a linear combination of the fundamental sequence:

$$w^{(r)} = \sum_{k=0}^{r-1} b_k U_{n-k+1}^{(r)},\ n \geqslant 0. \tag{3.3}$$

The first r of these relations (3.3) may be considered as a system of simultaneous equations in the b_k as unknowns; since the determinant of the system is

$$\begin{vmatrix} U_1^{(r)} & 0 & \dots & 0 \\ U_2^{(r)} & U_1^{(r)} & \dots & 0 \\ U_r^{(r)} & U_{r-1}^{(r)} & \dots & U_1^{(r)} \end{vmatrix} = 1,\ (U_1^{(r)} = 1),$$

the solution always exists, is unique, and can be expressed easily in determinant form. To obtain a simpler expression, we calculate for $n < r$ that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j P_{rj}\, w_{n-j}^{(r)} &= \sum_{j=0}^{n}\sum_{k=0}^{r-1}(-1)^j P_{rj}\, b_k U_{n-j-k+1}^{(r)} \\ &= \sum_{k=0}^{r-1} b_k \sum_{j=0}^{n}(-1)^j P_{rj}\, U_{n-k-j+1}^{(r)} \\ &= -b_n,\end{aligned}$$

since

$$\sum_{j=0}^{n}(-1)^j P_{rj}\, U_{n-k-j+1}^{r} = P_{r0}\,\delta_{nk},$$

where δ_{nk} is again the Kronecker delta; this follows from the facts that

$$U_n^{(r)} = 0 \quad \text{if } n \leqslant 0,$$

$$\sum_{j=0}^{r}(-1)^j P_{rj}\, U_{n-j+1}^{(r)} = 0 \quad \text{if } n > 0,$$

and

$$U_1^{(r)} = 1.$$

Thus,

$$\begin{aligned}w_n^{(r)} &= -\sum_{k=0}^{r-1}\sum_{i=0}^{k}(-1)^{i+1} P_{ri}\, w_{k-i}^{(r)} U_{n-k+1}^{(r)} \\ &= -\sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1}(-1)^{k-j+1} P_{r,k-j} U_{n-k+1}^{(r)}\right) w^{(r)}.\end{aligned}$$

That is, $w_n^{(r)}$ depends on the initial values $w_0^{(r)}$, $w_1^{(r)}$, ..., $w_{r-1}^{(r)}$, and $U_n^{(r)}$, and so the properties of $w_n^{(r)}$ depend on the $U_n^{(r)}$.

We now seek the rank of $U_n^{(r)}$ instead of $w_n^{(r)}$; this will be a generalization of Subba Rao rather than Horadam. From Eq. (3.1), we have that

$$\alpha_{r1}^{n} = \frac{1}{r}\, U_n^{(r)} \sum_{k=1}^{r} D^k \Big/ \omega^k.$$

So

$$\alpha_{r1}^{n} > \frac{1}{r}\, U_n^{(r)} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k}$$

and

$$\alpha_{r1}^{n} < \frac{1}{r}\, U_n^{(r)} \sum_{k=1}^{r} D^k.$$

Thus,

$$n + 1 < \underline{\log}\; U_n^{(r)} \left(\alpha_{r1} \sum_{k=1}^{r} D^k\right) \Big/ r$$

and

$$n + 1 > \underline{\log}\; U_n^{(r)} \left(\alpha_{r1} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k}\right) \Big/ r,$$

which yield:

Theorem B

$$\underline{\log}\; U_n^{(r)} + \underline{\log}\left(\alpha_{r1} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k} \Big/ r\right) < n + 1 < \underline{\log}\; U_n^{(r)} + \underline{\log}\left(\alpha_{r1} \sum_{k=1}^{r} D^k \Big/ r\right),$$

and this gives the range within which the rank n of $U_n^{(r)}$ lies.

For example, when $r = 2$, $D = d$, $d^2 = 5$, $\omega = -1$, $P_{21} = -P_{22} = 1$, $\alpha_{21} \doteqdot 1.6$, $\alpha_{22} \doteqdot -0.6$, we get for the Fibonacci number $F_3^{(2)} = 2$ that

$$\frac{\log 2}{\log 1.6} + \frac{\log 2.9}{\log 1.6} < 3 + 1 < \frac{\log 2}{\log 1.6} + \frac{\log 5.8}{\log 1.6} \quad \text{or} \quad 3.7 < 5 < 5.3.$$

This is not quite as good as Subba Rao's result, because there is just one number in the corresponding range for his result, but we do have an acceptable range.

Thus, in Theorem A we have generalized Horadam's result, and in Theorem B we have generalized Subba Rao's result; we have also established a link between the more generalized sequence in Theorem A and the fundamental generalized sequence in Theorem B.

REFERENCES

1. A. F. Horadam. "Generalizations of Two Theorems of K. Subba Rao." *Bulletin of the Calcutta Mathematical Society* 58 (1966):23-29.
2. Dov Jarden. *Recurring Sequences*. Riveon Lematematika, 1966, p. 107.
3. K. Subba Rao. "Some Properties of Fibonacci Numbers—I." *Bulletin of the Calcutta Mathematical Society* 46 (1954):253-57.

◆◇◆◇◆