INFINITE CLASSES OF SEQUENCE-GENERATED CIRCLES

A. G. SHANNON
The N.S.W. Institute of Technology, Sydney, 2007, Australia

A. F. HORADAM
University of New England, Armidale, 2351, Australia

and

GERALD E. BERGUM
South Dakota State University, Brookings, SD 57007-1297

(Submitted November 1982)

1. INTRODUCTION

In a previously published paper on the geometry of a generalized Simson's formula, Horadam [2] considered the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of consecutive elements of a generalized Fibonacci sequence. A Simson's formula as generalized by Horadam [1] was employed in obtaining the loci.

In this paper, we also utilize the same Simson's formula to develop a generalized "Fibonacci circle"; that is, we show how the locus of a point generated by three consecutive elements of the generalized Fibonacci sequence $\{w_n\}$, defined below, approximates a circle *for large* n, subject to special restrictions.

We define the sequence $\{w_n\}$ by

$$w_{n+2} = pw_{n+1} - qw_n, \quad w_0 = a, \; w_1 = b, \tag{1.1}$$

where a, b, p, and q belong to some number system but are usually thought of as integers [1].

It is common knowledge that the terms of $\{w_n\}$ are related to the roots of the equation

$$\lambda^2 - p\lambda + q = 0. \tag{1.2}$$

We denote the roots by

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

and assume throughout the remainder of this paper that

$$\begin{aligned} &\text{(a)} \quad p^2 > 4q, \\ &\text{(b)} \quad p^2 - 4q \neq t^2 \\ &\text{(c)} \quad |q| \leqslant 1 \\ &\text{(d)} \quad \alpha < 1 + \sqrt{2} \\ &\text{(e)} \quad \{w_n\} \text{ is strictly increasing.} \end{aligned} \tag{1.3}$$

Now $\alpha\beta = q$, so parts (c) and (d) of (1.3) tell us that $|\beta| < 1$. Therefore, from Horadam [1, 3.1], we know

$$\lim_{n \to \infty} \frac{w_n}{w_{n-1}} = \alpha. \tag{1.4}$$

In closing, we observe that part (b) of (1.3) guarantees that $p \neq 1 + q$, which is enough to show that $\alpha \neq 1$. Part (b) with (e) is also enough to show that

$$\lim_{n \to \infty} w_n = \infty. \tag{1.5}$$

2. PRELIMINARIES

Let k, ℓ, and m be three consecutive terms of $\{w_n\}$ with $k = w_n$.

Since w_n is strictly increasing and $w_n \to \infty$, we may as well consider throughout the rest of the paper only those terms of w_n that are greater than 0.

From [1, 4.3 & 1.9], we know that

$$\begin{aligned} \ell^2 - mk &= -eq^n && (2.1)\\ &= -(pab - qa^2 - b^2)q^n \\ &= (w_1^2 - w_0 w_2)q^n && \text{by (1.1)} \\ &< M && \text{by (1.3), part (c)} \end{aligned}$$

for some positive integer M. We also have

$$\lim_{n \to \infty}(\ell - k) = \lim_{n \to \infty} k\left(\frac{\ell}{k} - 1\right) = \infty, \tag{2.2}$$

by (1.4) and (1.5). Hence, for n sufficiently large,

$$\ell^2 - mk < \ell - k \tag{2.3}$$

or, with r as the midpoint of $\frac{\ell - 1}{k}$ and $\frac{m - 1}{\ell}$,

$$\frac{\ell - 1}{k} < r = \frac{\ell^2 + km - \ell - k}{2k\ell} < \frac{m - 1}{\ell}. \tag{2.4}$$

From (2.4), we immediately have

$$\ell - rk < 1 < m - r\ell. \tag{2.5}$$

Using (2.1), (2.4), and (1.4), we see that

$$\lim_{n \to \infty}(\ell - rk) = \lim_{n \to \infty} \frac{\ell^2 - km + k + \ell}{2\ell} = \frac{\alpha + 1}{2\alpha} \tag{2.6}$$

and

$$\lim_{n \to \infty}(m - r\ell) = \lim_{n \to \infty} \frac{km - \ell^2 + \ell + k}{2k} = \frac{\alpha + 1}{2}. \tag{2.7}$$

Since $\alpha > 0$, we can now strengthen (2.5) using (2.6) to

$$0 < \ell - rk < 1 < m - r\ell, \; n \text{ sufficiently large}. \tag{2.8}$$

Another obvious conclusion of (2.6) and (2.7) is

$$\lim_{n \to \infty} \frac{m - r\ell}{\ell - rk} = \alpha. \tag{2.9}$$

In conclusion, using (2.6) and (2.7) with part (d) of (1.3), let us observe that

$$\lim_{n \to \infty}(\ell - rk + 1 - m + r\ell) = \frac{1 + 2\alpha - \alpha^2}{2\alpha} > 0 \tag{2.10}$$

so that for n sufficiently large

$$\ell - rk + 1 > m - r\ell. \tag{2.11}$$

3. THE GEOMETRY

Throughout this section, we assume n is sufficiently large. We let

$$\begin{aligned} AB &= 1 \\ QA &= \ell - rk \\ QB &= m - r\ell \end{aligned} \tag{3.1}$$

and locate the origin of our system by setting

$$OA = 1/(\alpha^2 - 1) \tag{3.2}$$

and by extending BA to O.

We let D be the foot of the perpendicular form Q to OB. By (2.8) and (2.11) this construction is legitimate and gives us the triangle QAB (see Figure 1).

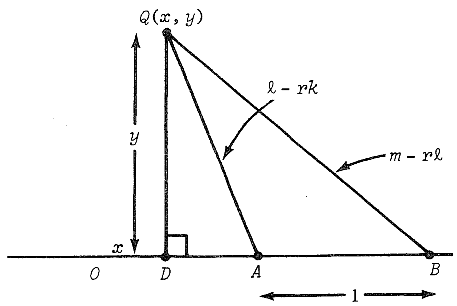

FIGURE 1

Now,

$$\begin{aligned} \text{area } QAB &= \frac{1}{2}DQ \\ &= \sqrt{(s(s - QB)(s - QA)(s - AB))} \end{aligned} \tag{3.3}$$

where s is the semi-perimeter of the triangle QAB.

For notational convenience, let

$$QA = u. \tag{3.4}$$

Then, for sufficiently large n, for which

$$QB = \alpha \cdot QA = \alpha u, \text{ by (2.9), (3.4)} \tag{3.5}$$

we have

$$s = \frac{1}{2}(\alpha u + u + 1), \text{ by (3.1), (3.4), (3.5)} \tag{3.6}$$

and so

$$\begin{aligned} 4DQ^2 &= (\alpha u + u + 1)(-\alpha u + u + 1)(\alpha u - u + 1)(\alpha u + u - 1), \\ &\qquad \text{by (3.1), (3.3), (3.4), (3.5), (3.6)} \\ &= ((\alpha u + u)^2 - 1)(1 - (\alpha u - u)^2) \\ &= 2u^2(\alpha^2 + 1) - 1 - u^4(\alpha^2 - 1)^2. \end{aligned} \tag{3.7}$$

Then,

$$\begin{aligned} 4DA^2 &= 4QA^2 - 4DQ^2 \quad \text{by the Pathagorean Theorem} \\ &= -2u^2(\alpha^2 - 1) + 1 + u^4(\alpha^2 - 1)^2, \text{ by (3.4), (3.7)} \\ &= (u^2(\alpha^2 - 1) - 1)^2. \end{aligned}$$

Whence

$$2DA = u^2(\alpha^2 - 1) - 1. \tag{3.8}$$

Now OD and DQ are the x- and y-coordinates, respectively, of Q, so that

$$\begin{aligned} x^2 + y^2 &= OD^2 + DQ^2 \\ &= (OA - DA)^2 + DQ^2 \\ &= OA^2 + DA^2 + DQ^2 - 2OA \cdot DA \\ &= OA^2 + QA^2 - OA(2DA) \quad \text{by the Pathagorean Theorem} \\ &= \frac{1}{(\alpha^2 - 1)^2} + u^2 - \frac{1}{(\alpha^2 - 1)}(u^2(\alpha^2 - 1) - 1), \\ &\qquad \text{by (3.2), (3.4), (3.8),} \\ &= \frac{1}{(\alpha^2 - 1)^2} + \frac{1}{\alpha^2 - 1} \\ &= \frac{\alpha^2}{(\alpha^2 - 1)^2}. \end{aligned}$$

That is,

$$x^2 + y^2 = \left(\frac{\alpha}{\alpha^2 - 1}\right)^2. \tag{3.9}$$

The locus of Q as n increases is, therefore, a circle with center 0 and radius $\alpha/(\alpha^2 - 1)$.

As p, q (and, consequently, α) vary, the corresponding sequences clearly generate an infinite set of concentric circles.

4. FIBONACCI-TYPE CIRCLES

For the sequence of ordinary Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, ..., we have

$$p = -q = 1, \; \alpha^2 = \alpha + 1, \text{ and } \alpha = \frac{1}{2}(1 + \sqrt{5}),$$

so the circle given by (3.9) becomes the unit circle.

Moreover, all sequences for which $p = -q = 1$ [and so for which $\alpha^2 = \alpha + 1$, $\alpha = (1/2)(1 + \sqrt{5})$], e.g., the Lucas sequence 2, 1, 3, 4, 7, 11, 18, 29, ..., give rise to this unit circle.

The following table illustrates the result for the Fibonacci numbers.

n	F_n	F_{n+1}	$x^2 + y^2$
2	1	2	.763932
3	2	3	.328550
4	3	5	.914537
5	5	8	.698798
6	8	13	1.003089
7	13	21	.878930
8	21	34	1.044630
9	34	55	.952913
10	55	89	1.029224
11	89	144	.981894
12	144	233	1.011208
13	233	377	.993066
14	377	610	1.004288
15	610	987	.997349
16	987	1597	1.001639
17	1597	2584	.998987
18	2584	4181	1.000626
19	4181	6765	.999613
20	6765	10946	1.000239
21	10946	17711	.999852
22	17711	28657	1.000091
23	28657	46368	.999944
24	46368	75025	1.000035
25	75025	121393	.999978
26	121393	196418	1.000013
27	196418	317811	.999992
28	317811	514229	1.000005
29	514229	832040	.999997
30	832040	1346269	1.000002

Gratitude is expressed to Wilson [3], whose Fibonacci circle, derived from five successive large Fibonacci numbers, was useful in the development of this theory.

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 3 (1965):161-76.
2. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *The Fibonacci Quarterly* 20, no. 2 (1982):164-68.
3. L. G. Wilson. "Proof of a Connection between the Circle and any Five Extremely Large Consecutive Terms on the Positive Side of the Fibonacci Sequence." Private publication, 1980.

◆◇◆◇◆