A NEW PERSPECTIVE TO THE GENERALIZATION OF THE FIBONACCI SEQUENCE

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I

Let the arbitrary real numbers a, b, c, and d be given. Construct two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ for which

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + \beta_n, \quad n \geqslant 0 \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n, \quad n \geqslant 0 \end{cases} \tag{1}$$

Clearly, if we set $a = b$ and $c = d$, then the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i(a, d)\}_{i=0}^{\infty}$. The first ten terms of the sequences defined in (1) are:

n	α_n	β_n
0	a	b
1	c	d
2	$b + d$	$a + c$
3	$a + c + d$	$b + c + d$
4	$a + b + 2c + d$	$a + b + c + 2d$
5	$a + 2b + 2c + 3d$	$2a + b + 3c + 2d$
6	$3a + 2b + 4c + 4d$	$2a + 3b + 4c + 4d$
7	$4a + 4b + 7c + 6d$	$4a + 4b + 6c + 7d$
8	$6a + 7b + 10c + 11d$	$7a + 6b + 11c + 10d$
9	$11a + 10b + 17c + 17d$	$10a + 11b + 17c + 17d$

A careful examination of the corresponding terms in each column leads one immediately to

Theorem 1.1

(a) $\alpha_{3n} + \beta_0 = \beta_{3n} + \alpha_0, \quad n \geqslant 0$

(b) $\alpha_{3n+1} + \beta_1 = \beta_{3n+1} + \alpha_1, \quad n \geqslant 0$

(c) $\alpha_{3n+2} + \alpha_0 + \alpha_1 = \beta_{3n+2} + \beta_0 + \beta_1, \quad n \geqslant 0$

Proof of (a): The statement is obviously true if $n = 0$. Assume the statement is true for some integer $n \geqslant 1$. Then

$\alpha_{3n+3} + \beta_0 = \beta_{3n+2} + \beta_{3n+1} + \beta_0$ by (1)

(continued)

$$\begin{aligned}
&= \alpha_{3n+1} + \alpha_{3n} + \beta_{3n+1} + \beta_0 && \text{by (1)}\\
&= \alpha_{3n+1} + \beta_{3n} + \beta_{3n+1} + \alpha_0 && \text{by induction hypothesis}\\
&= \alpha_{3n+1} + \alpha_{3n+2} + \alpha_0 && \text{by (1)}\\
&= \beta_{3n+3} + \alpha_0 && \text{by (1).}
\end{aligned}$$

Hence, the statement is true for all integers $n \geqslant 0$. Similar proofs can be given for parts (b) and (c).

Adding the first n terms of each sequence $\{\alpha_i\}$ and $\{\beta_i\}$ yields a result similar to that obtained by adding the first n Fibonacci numbers. That is,

Theorem 1.2. For all integers $k \geqslant 0$, we have:

(a) $\alpha_{3k+2} = \sum_{i=0}^{3k} \beta_i + \beta_1$ (d) $\beta_{3k+2} = \sum_{i=0}^{3k} \alpha_i + \alpha_1$

(b) $\alpha_{3k+3} = \sum_{i=0}^{3k+1} \alpha_i + \beta_1$ (e) $\beta_{3k+3} = \sum_{i=0}^{3k+1} \beta_i + \alpha_1$

(c) $\alpha_{3k+4} = \sum_{i=0}^{3k+2} \beta_i + \alpha_1$ (f) $\beta_{3k+4} = \sum_{i=0}^{3k+2} \alpha_i + \beta_1$

Because the proofs of each part are very similar, we give only a proof of part (e).

Proof of (e): If $k = 0$ the statement is obviously true, since

$$\sum_{i=0}^{1} \beta_i + \alpha_1 = \beta_0 + \beta_1 + \alpha_1 = \alpha_2 + \alpha_1 = \beta_3.$$

Assume (e) is true for some integer $k \geqslant 1$, then

$$\begin{aligned}
\beta_{3k+6} &= \alpha_{3k+5} + \alpha_{3k+4} && \text{by (1)}\\
&= \beta_{3k+4} + \beta_{3k+3} + \alpha_{3k+4} && \text{by (1)}\\
&= \beta_{3k+4} + \sum_{i=0}^{3k+1} \beta_i + \alpha_1 + \beta_{3k+3} + \beta_{3k+2} && \text{by induction hypothesis and (1)}\\
&= \sum_{i=0}^{3k+4} \beta_i + \alpha_1.
\end{aligned}$$

Hence, (e) is true for all integers $k \geqslant 0$.

Adding the first n terms with even or odd subscripts for each sequence $\{\alpha_i\}$ and $\{\beta_i\}$, we obtain more results which are similar to those obtained when one adds the first n terms of the Fibonacci sequence with even or odd subscripts. That is,

Theorem 1.3. For all integers $k \geqslant 0$, we have:

(a) $\alpha_{6k+5} = \sum_{i=0}^{3k+2} \beta_{2i} - \alpha_0 + \beta_1$ (c) $\alpha_{6k+7} = \sum_{i=0}^{3k+3} \beta_{2i} - \beta_0 + \alpha_1$

(b) $\alpha_{6k+6} = \sum_{i=1}^{3k+3} \beta_{2i-1} + \alpha_0$ (d) $\alpha_{6k+8} = \sum_{i=1}^{3k+4} \beta_{2i-1} + \beta_0$

(e) $\alpha_{6k+9} = \sum_{i=0}^{3k+4} \beta_{2i} - \beta_0 + \beta_1$

(f) $\alpha_{6k+10} = \sum_{i=1}^{3k+5} \beta_{2i-1} + \alpha_0 + \alpha_1 - \beta_1$

(g) $\beta_{6k+5} = \sum_{i=0}^{3k+2} \alpha_{2i} - \beta_0 + \alpha_1$

(h) $\beta_{6k+6} = \sum_{i=1}^{3k+3} \alpha_{2i-1} + \beta_0$

(i) $\beta_{6k+7} = \sum_{i=0}^{3k+3} \alpha_{2i} - \alpha_0 + \beta_1$

(j) $\beta_{6k+8} = \sum_{i=1}^{3k+4} \alpha_{2i-1} + \alpha_0$

(k) $\beta_{6k+9} = \sum_{i=0}^{3k+4} \alpha_{2i} - \alpha_0 + \alpha_1$

(l) $\beta_{6k+10} = \sum_{i=1}^{3k+5} \alpha_{2i-1} + \beta_0 - \alpha_1 + \beta_1$

Proof of (g): If $k = 0$ the statement is obviously true, since

$$\sum_{i=0}^{2} \alpha_{2i} - \beta_0 + \alpha_1 = \alpha_0 + \alpha_2 + \alpha_4 - \beta_0 + \alpha_1 = 2a + b + 3c + 2d = \beta_5.$$

Assume (g) is true for some integer $k \geqslant 1$, then

$$\begin{aligned}
\beta_{6k+11} &= \alpha_{6k+10} + \alpha_{6k+9} && \text{by (1)} \\
&= \alpha_{6k+10} + \beta_{6k+9} + \alpha_0 - \beta_0 && \text{by Theorem 1.1, part (a)} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \alpha_{6k+7} + \alpha_0 - \beta_0 && \text{by (1)} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \beta_{6k+6} + \sum_{i=0}^{3k+2} \alpha_{2i} + \alpha_1 + \alpha_0 - 2\beta_0 && \text{by (1) and induction hypothesis} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \sum_{i=0}^{3k+3} \alpha_{2i} + \alpha_1 - \beta_0 && \text{by Theorem 1.1, part (a)} \\
&= \sum_{i=0}^{3k+5} \alpha_{2i} + \alpha_1 - \beta_0.
\end{aligned}$$

Hence, (g) is true for all integers $k \geqslant 0$. A similar proof can be given for each of the remaining eleven parts of the theorem.

The following result is an interesting relationship which follows immediately from Theorems 1.1 and 1.2. Therefore, the proofs are omitted.

Theorem 1.4. If $k \geqslant 0$, then

(a) $\sum_{i=0}^{3k} (\alpha_i - \beta_i) = \alpha_0 - \beta_0$

(b) $\sum_{i=0}^{3k+1} (\alpha_i - \beta_i) = \beta_2 - \alpha_2$

(c) $\sum_{i=0}^{3k+2} (\alpha_i - \beta_i) = 0.$

As one might suspect, there should be a relationship between the new sequence and the Fibonacci numbers. The next theorem establishes one of these relationships.

Theorem 1.5. If $n \geqslant 0$, then

$$\alpha_{n+2} + \beta_{n+2} = F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1).$$

Proof: The statement is obviously true if $n = 0$ and $n = 1$. Assume that the statement is true for all integers less than or equal to some $n \geqslant 2$. Then

$$\begin{aligned}\alpha_{n+3} + \beta_{n+3} &= \beta_{n+2} + \beta_{n+1} + \alpha_{n+2} + \alpha_{n+1} && \text{by (1)}\\ &= F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_n(\alpha_0 + \beta_0)\\ &\qquad + F_{n+1}(\alpha_1 + \beta_1) && \text{by induction hypothesis}\\ &= F_{n+2}(\alpha_0 + \beta_0) + F_{n+3}(\alpha_1 + \beta_1).\end{aligned}$$

Hence, the statement is true for all integers $n \geqslant 0$.

At this point, one could continue to establish properties for the two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ which are similar to those of the Fibonacci sequence. However, we have chosen another route.

II

Express the members of the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$, when $n \geqslant 0$, as follows:

$$\left.\begin{aligned}\alpha_n &= \Gamma_n^1 a + \Gamma_n^2 b + \Gamma_n^3 c + \Gamma_n^4 d\\ \beta_n &= \delta_n^1 a + \delta_n^2 b + \delta_n^3 c + \delta_n^4 d\end{aligned}\right\} \qquad (2)$$

In this way we obtain the eight sequences $\{\Gamma_i^j\}_{i=0}^{\infty}$, $\{\delta_i^j\}_{i=0}^{\infty}$, $(j = 1,2,3,4)$. The purpose of this section is to show how these eight sequences are related to each other and to the Fibonacci numbers with the major intent of finding a direct formula for calculating α_n and β_n for any n.

Theorem 2.1 establishes a relationship between these eight sequences and the Fibonacci numbers.

Theorem 2.1

(a) $\Gamma_n^1 + \delta_n^1 = F_{n-1}, \quad n \geqslant 0$

(b) $\Gamma_n^2 + \delta_n^2 = F_{n-1}, \quad n \geqslant 0$

(c) $\Gamma_n^3 + \delta_n^3 = F_n, \quad n \geqslant 0$

(d) $\Gamma_n^4 + \delta_n^4 = F_n, \quad n \geqslant 0.$

Proof of (a): This is obviously true if $n = 0$ and 1, since

$$\Gamma_0^1 + \delta_0^1 = 1 + 0 = F_{-1} \quad \text{and} \quad \Gamma_1^1 + \delta_1^1 = 0 + 0 = 0 = F_0.$$

Assume this is true for all integers less than or equal to some integer $n \geqslant 2$. Then

$$\Gamma^1_{n+1} + \delta^1_{n+1} = \delta^1_n + \delta^1_{n-1} + \Gamma^1_n + \Gamma^1_{n-1} = F_{n-1} + F_{n-2} = F_n,$$

and (a) is true for all integers $n \geqslant 0$. Similarly, one can prove parts (b), (c), and (d).

The next step is to show how the above eight sequences are related to each other.

Theorem 2.2. If $k \geqslant 0$, then

(a) $\Gamma^1_{3k} = \delta^1_{3k} + 1$

(b) $\Gamma^1_{3k+1} = \delta^1_{3k+1}$

(c) $\Gamma^1_{3k+2} = \delta^1_{3k+2} - 1$

(d) $\Gamma^2_{3k} = \delta^2_{3k} - 1$

(e) $\Gamma^2_{3k+1} = \delta^2_{3k+1}$

(f) $\Gamma^2_{3k+2} = \delta^2_{3k+2} + 1$

(g) $\Gamma^3_{3k} = \delta^3_{3k}$

(h) $\Gamma^3_{3k+1} = \delta^3_{3k+1} + 1$

(i) $\Gamma^3_{3k+2} = \delta^3_{3k+2} - 1$

(j) $\Gamma^4_{3k} = \delta^4_{3k}$

(k) $\Gamma^4_{3k+1} = \delta^4_{3k+1} - 1$

(l) $\Gamma^4_{3k+2} = \delta^4_{3k+2} + 1$

Proof of (j): It is obvious that (j) is true if $k = 0$, since $\Gamma^4_0 = \delta^4_0 = 0$. Assume the statement is true for some integer $k \geqslant 1$. Then

$$\begin{aligned}\Gamma^4_{3k+3} &= \delta^4_{3k+2} + \delta^4_{3k+1} && \text{by (1)}\\ &= \Gamma^4_{3k+1} + \Gamma^4_{3k} + \delta^4_{3k+1} && \text{by (1)}\\ &= \Gamma^4_{3k+1} + \delta^4_{3k} + \delta^4_{3k+1} && \text{by induction hypothesis}\\ &= \Gamma^4_{3k+1} + \Gamma^4_{3k+2} = \delta^4_{3k+3} && \text{by (1)}\end{aligned}$$

and the statement is proved. The remaining parts are proved in a similar way.

We now show

Theorem 2.3. If $n \geqslant 0$, then

(a) $\Gamma^1_n + \Gamma^2_n = \delta^1_n + \delta^2_n$

(b) $\Gamma^3_n + \Gamma^4_n = \delta^3_n + \delta^4_n$

Proof of (a): This is obviously true if $n = 0$ and $n = 1$. Assume true for all integers less than or equal to some integer $n \geqslant 2$. Then

$$\begin{aligned}\Gamma^1_{n+1} + \Gamma^2_{n+1} &= \delta^1_n + \delta^1_{n-1} + \delta^2_n + \delta^2_{n-1} && \text{by (1)}\\ &= \Gamma^1_n + \Gamma^2_n + \Gamma^1_{n-1} + \Gamma^2_{n-1} && \text{by induction hypothesis}\\ &= \delta^1_{n+1} + \delta^2_{n+1} && \text{by (1)}\end{aligned}$$

Similarly, one can prove part (b).

Before stating and proving our main result for this section, we need the following three theorems.

Theorem 2.4. If $n \geqslant 0$, then

(a) $\delta_n^1 = \Gamma_n^2$

(b) $\delta_n^2 = \Gamma_n^1$

(c) $\delta_n^3 = \Gamma_n^4$

(d) $\delta_n^4 = \Gamma_n^3$

(e) $\Gamma_n^3 = \Gamma_{n+1}^2$

(f) $\Gamma_n^4 = \Gamma_{n+1}^1$

(g) $\delta_n^3 = \delta_{n+1}^2$

(h) $\delta_n^4 = \delta_{n+1}^1$

Proof of (a): The statement is trivially true for $n = 0, 1, 2$, so assume it is true for all integers less than or equal to n where $n \geqslant 3$. Then

$$\begin{aligned}\delta_{n+1}^1 &= \Gamma_n^1 + \Gamma_{n-1}^1 && \text{by (1)}\\ &= \delta_{n-1}^1 + \delta_{n-2}^1 + \delta_{n-2}^1 + \delta_{n-3}^1 && \text{by (1)}\\ &= \Gamma_{n-1}^2 + \Gamma_{n-2}^2 + \Gamma_{n-2}^2 + \Gamma_{n-3}^2 && \text{by induction hypothesis}\end{aligned}$$

Two applications of (1) will complete the proof of part (a) of the theorem. The other parts are proved by similar arguments.

From Theorems 2.1 and 2.4, we have the following.

Theorem 2.5

(a) $\Gamma_n^1 + \Gamma_n^2 = \delta_n^1 + \delta_n^2 = F_{n-1} \quad (n \geqslant 0)$

(b) $\Gamma_n^3 + \Gamma_n^4 = \delta_n^3 + \delta_n^4 = F_n \quad (n \geqslant 0)$

Finally, we have the following statement.

Theorem 2.6. If $n \geqslant 2$, then

(a) $\Gamma_n^1 = \Gamma_{n-1}^1 + \Gamma_{n-2}^1 + 3\left[\frac{n}{3}\right] - n + 1$

(b) $\Gamma_n^2 = \Gamma_{n-1}^2 + \Gamma_{n-2}^2 + n - 3\left[\frac{n}{3}\right] - 1$

(c) $\Gamma_n^1 = \Gamma_n^2 + 3\left[\frac{n}{3}\right] - n + 1$

(d) $\Gamma_n^3 = \Gamma_{n-1}^3 + \Gamma_{n-2}^3 + n - 3\left[\frac{n+1}{3}\right]$

(e) $\Gamma_n^4 = \Gamma_{n-1}^4 + \Gamma_{n-2}^4 + 3\left[\frac{n+1}{3}\right] - n$

(f) $\Gamma_n^3 = \Gamma_n^4 + n - 3\left[\frac{n+1}{3}\right]$

Proof of (a): The statement is obviously true if n equals 2 or 3. Assume the statement true for all integers less than or equal to $n \geqslant 4$. Then

$$\begin{aligned}\Gamma_{n+1}^1 &= \delta_n^1 + \delta_{n-1}^1 = \Gamma_n^2 + \Gamma_{n-1}^2 && \text{by (1) and Theorem 2.4, part (a)}\\ &= \delta_{n-1}^2 + \delta_{n-2}^2 + \delta_{n-2}^2 + \delta_{n-3}^2 && \text{by (1)}\\ &= \Gamma_{n-1}^1 + \Gamma_{n-2}^1 + \Gamma_{n-2}^1 + \Gamma_{n-3}^1 && \text{by Theorem 2.4, part (b)}\end{aligned}$$

(continued)

$$= \Gamma_n^1 - 3\left[\frac{n}{3}\right] + n - 1 + \Gamma_{n-1}^1 - 3\left[\frac{n-1}{3}\right] + n - 2 \quad \text{by induction hypothesis}$$
$$= \Gamma_n^1 + \Gamma_{n-1}^1 + 2n - 3 - 3\left[\frac{n}{3}\right] - 3\left[\frac{n-1}{3}\right]$$
$$= \Gamma_n^1 + \Gamma_{n-1}^1 + 2n - 3 + 3\left[\frac{n+1}{3}\right] - 3n + 3$$
$$= \Gamma_n^1 + \Gamma_{n-1}^1 + 3\left[\frac{n+1}{3}\right] - (n-1) + 1$$

and part (a) is proved. (It can be shown that $[(n+1)/3] + [n/3] + [(n-1)/3] = n - 1$, $n \geqslant 1$.) Similarly, one can prove parts (b), (d), and (e).

The proof of part (c) above follows directly from part (a) of Theorem 2.6, (1), and part (a) of Theorem 2.4. The proof of part (f) follows by a similar argument.

Adding the equations of part (a) of both Theorems 2.5 and 2.6, we have, for $n \geqslant 0$,

$$\Gamma_{n+2}^1 = \frac{1}{2}\left(F_{n+1} - \Gamma_{n+2}^2 + \Gamma_{n+1}^1 + \Gamma_n^1 + 3\left[\frac{n+2}{3}\right] - n - 1\right)$$
$$= \frac{1}{2}\left(F_{n+1} - \Gamma_{n+2}^2 + \delta_{n+2}^1 + 3\left[\frac{n+2}{3}\right] - n - 1\right) \quad \text{by (1)}$$
$$= \frac{1}{2}\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right) \quad \text{by (a) of Theorem 2.4}$$
$$= \delta_{n+2}^2 \quad \text{by (a) of Theorem 2.4}$$

Similarly, we have

$$\Gamma_{n+2}^2 = \frac{1}{2}\left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right) = \delta_{n+2}^1$$
$$\Gamma_{n+2}^3 = \frac{1}{2}\left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right) = \delta_{n+2}^4$$
$$\Gamma_{n+2}^4 = \frac{1}{2}\left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right) = \delta_{n+2}^3.$$

Substiting these four equations into (2), we have our

BASIC THEOREM. If $n \geqslant 0$, then

$$\alpha_{n+2} = \frac{1}{2}\left\{\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)a + \left(F_{n+1} + n + 1 - 3\left[\frac{n+2}{3}\right]\right)b + \left(F_{n+2} + n - 3\left[\frac{n}{3}\right] - 1\right)c + \left(F_{n+2} + 3\left[\frac{n}{3}\right] + 1 - n\right)d\right\}$$
$$= \frac{1}{2}\left\{(a+b)F_{n+1} + (c+d)F_{n+2} + \left(3\left[\frac{n+2}{3}\right] - n - 1\right)(a-b) + \left(n - 3\left[\frac{n}{3}\right] - 2\right)(c-d)\right\};$$

$$\beta_{n+2} = \frac{1}{2}\left\{\left(F_{n+1} + n + 1 - 3\left[\frac{n+2}{3}\right]\right)a + \left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)b + \left(F_{n+2} + 3\left[\frac{n}{3}\right] + 1 - n\right)c + \left(F_{n+2} + n - 3\left[\frac{n}{3}\right] - 1\right)d\right\}$$

(continued)

$$= \frac{1}{2}\left\{(a + b)F_{n+1} + (c + d)F_{n+2} + \left(3\left[\frac{n + 2}{3}\right] - n - 1\right)(b - a) + \left(n - 3\left[\frac{n}{3}\right] - 1\right)(d - c)\right\}.$$

III

The sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ can also be expressed as follows [similarly to (1)]:

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \alpha_{n+1} + \alpha_n \\ \beta_{n+2} &= \beta_{n+1} + \beta_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{3}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \beta_{n+1} + \alpha_n \\ \beta_{n+2} &= \alpha_{n+1} + \beta_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{4}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \alpha_{n+1} + \beta_n \\ \beta_{n+2} &= \beta_{n+1} + \alpha_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{5}$$

The sequences (3) are actually two independent Fibonacci sequences of the form $\{F_i(a, c)\}_{i=0}^{\infty}$ and $\{F_i(b, d)\}_{i=0}^{\infty}$. It is easily seen that the sequences (4) can be expressed through the sequences $\{F_i(a, d)\}_{i=0}^{\infty}$ and $\{F_i(b, c)\}_{i=0}^{\infty}$, namely, $\alpha_{2n} = F_{2n}(a, d)$, $\alpha_{2n+1} = F_{2n+1}(b, c)$, $\beta_{2n} = F_{2n}(b, c)$, $\beta_{2n+1} = F(a, d)$, $n \geqslant 1$.

In the case of (5), two sequences are introduced whose members are related similarly to those discussed in I and II. Therefore, we shall discuss them no further here.

Numerous similar pairs of sequences can be constructed. However, the ones introduced here stand most closely to the very spirit of the Fibonacci sequence and its generalization rules.

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