ON LUCAS FUNDAMENTAL FUNCTIONS AND CHEBYCHEV POLYNOMIAL SEQUENCES

S. PETHE
University of Malaya, Kuala Lumpur, Malaysia
(Submitted June 1983)

1. INTRODUCTION

Let $\{U_n(p, q)\}$ be the sequence of fundamental functions defined by Lucas [2] as follows:

$$U_{n+2} = pU_{n+1} - qU_n \qquad (n \geqslant 0)$$

with initial values $U_0 = 0$, $U_1 = 1$. Further, let $\{S_n(x)\}$ and $\{T_n(x)\}$ denote the Chebychev polynomial sequences of the first and second kind, respectively. In [5], formulas were obtained for

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+j}}{(3n+j)!}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+j}(x)}{(3n+j)!}, \quad \text{and} \sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+j}(x)}{(3n+j)!}, \quad j = 0, 1, 2.$$

As mentioned in [5, Remark 4], we generalize the above formulas in this paper to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!}, \quad j = 0, 1, \ldots, p-1,$$

and similar formulas for $\{S_n(x)\}$ and $\{T_n(x)\}$.

2. PRELIMINARIES

The generalized circular functions are defined as follows. For any positive integer r,

$$M_{r,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1,$$

and

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1.$$

Note that $M_{1,0}(t) = e^{-t}$, $M_{2,0}(t) = \cos t$, $M_{2,1}(t) = \sin t$, and $N_{1,0}(t) = e^t$, $N_{2,0}(t) = \cosh t$, $N_{2,0}(t) = \sinh t$.

The notation and some of the results presented here are found in Pethe and Sharma [4].

Following Barakat [1] and Walton [7], we define generalized trigonometric and hyperbolic functions of any square matrix X by

$$M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n X^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1,$$

and

$$N_{r,j}(X) = \sum_{n=0}^{\infty} \frac{X^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1.$$

Lemma 1. Let X be a 2×2 matrix given by

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Let tr $X = p$ and det $X = q$. Then, for any integer n,

$$X^n = U_n X - qU_{n-1}I,$$

where U_n is the n^{th} fundamental function and I the unit matrix of order 2.

This is proved in [1].

Lemma 2. We have, for a positive integer r and $j = 0, 1, \ldots, r - 1$,

$$M_{r,j}(x + y) = \sum_{k=0}^{j} M_{r,k}(x)M_{r,j-k}(y) - \sum_{k=j+1}^{r-1} M_{r,k}(x)M_{r,r+j-k}(y).$$

This is proved in [3].

Lemma 3. Let r be a positive integer, and $j = 0, 1, \ldots, r - 1$. Then:

a. For even r,

$$M_{r,j}(x) + M_{r,j}(-x) = \begin{cases} 2M_{r,j}(x), & j \text{ even} \\ 0, & j \text{ odd}, \end{cases} \tag{2.1}$$

and

$$M_{r,j}(x) - M_{r,j}(-x) = \begin{cases} 0, & j \text{ even} \\ 2M_{r,j}(x), & j \text{ odd}. \end{cases} \tag{2.2}$$

b. For odd r,

$$M_{r,j}(x) + M_{r,j}(-x) = \begin{cases} 2N_{2r,j}(x), & j \text{ even} \\ -2N_{2r,r+j}(x), & j \text{ odd}, \end{cases} \tag{2.3}$$

and

$$M_{r,j}(x) - M_{r,j}(-x) = \begin{cases} -2N_{2r,r+j}(x), & j \text{ even} \\ 2N_{2r,j}(x), & j \text{ odd}. \end{cases} \tag{2.4}$$

Proof: We prove (2.1) and (2.4). The proofs of (2.2) and (2.3) are similar.

Let r be even. Now,

$$M_{r,j}(x) + M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr + j)!}(1 + (-1)^{nr+j}). \tag{2.5}$$

Since r is even, $(-1)^{nr+j} = (-1)^j$. Hence (2.5) becomes

$$M_{r,j}(x) + M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr + j)!}(1 + (-1)^j) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr + j)!}, & j \text{ even} \\ 0, & j \text{ odd}, \end{cases}$$

(continued)

$$= \begin{cases} 2M_{r,j}(x), & j \text{ even} \\ 0, & j \text{ odd}, \end{cases}$$

which proves (2.1).

Now, let r be odd. Then

$$M_{r,j}(x) - M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 - (-1)^{nr+j}). \tag{2.6}$$

Since r is odd, $(-1)^{nr+j} = (-1)^{n(r-1)+n+j} = (-1)^{n+j}$; therefore, (2.6) becomes

$$M_{r,j}(x) - M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 - (-1)^{n+j})$$

$$= \begin{cases} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr+j)!}, & j \text{ even} \\ \displaystyle\sum_{n=0,2,\ldots}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr+j)!}, & j \text{ odd}, \end{cases}$$

$$= \begin{cases} \displaystyle -2\sum_{n=0}^{\infty} \frac{x^{2nr+r+j}}{(2nr+r+j)!}, & j \text{ even} \\ \displaystyle 2\sum_{n=0}^{\infty} \frac{x^{2nr+j}}{(2nr+j)!}, & j \text{ odd}, \end{cases}$$

$$= \begin{cases} -2N_{2r,r+j}(x), & j \text{ even} \\ 2N_{2r,j}(x), & j \text{ odd}, \end{cases}$$

which proves (2.4).

Lemma 4. We have for $j = 0, 1, \ldots, 2r-1$ and $i = \sqrt{-1}$,

$$\text{a.}\quad M_{2r,j}(ix) = \begin{cases} (-1)^{j/2} M_{2r,j}(x), & r \text{ even} \\ (-1)^{j/2} N_{2r,j}(x), & r \text{ odd}, \end{cases} \tag{2.7}$$

$$\text{b.}\quad N_{2r,j}(ix) = \begin{cases} (-1)^{j/2} N_{2r,j}(x), & r \text{ even} \\ (-1)^{j/2} M_{2r,j}(x), & r \text{ odd}. \end{cases} \tag{2.8}$$

Proof: By definition,

$$M_{2r,j}(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n i^{2nr+j} x^{2nr+j}}{(2nr+j)!}. \tag{2.9}$$

Now

$$(i)^{2nr+j} = \begin{cases} (i^4)^{nr/2}(i)^j, & r \text{ even} \\ (i^4)^{\frac{1}{2}n(r-1)}(i)^{2n}(i)^j, & r \text{ odd}, \end{cases}$$

so that

$$(i)^{2nr+j} = \begin{cases} (-1)^{j/2}, & r \text{ even} \\ (-1)^{n+j/2}, & r \text{ odd.} \end{cases} \tag{2.10}$$

Using (2.10) in (2.9), we obtain

$$M_{2r,j}(ix) = \begin{cases} (-1)^{j/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2nr+j}}{(2nr+j)!}, & r \text{ even} \\ (-1)^{j/2} \sum_{n=0}^{\infty} \frac{(-1)^{2n} x^{2nr+j}}{(2nr+j)!}, & r \text{ odd,} \end{cases}$$

which proves (2.7). We can prove (2.8) in a similar manner.

3. SUMMATION FORMULAS FOR LUCAS FUNDAMENTAL FUNCTIONS

We shall now prove

Theorem 1. a. For even r and $j = 0, 1, \ldots, r - 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \frac{2}{\delta}\left[\sum_{k=0}^{[\frac{1}{2}(j-1)]} M_{r,2k+m}(p/2)M_{r,\alpha}(\delta/2) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} M_{r,2k+m}(p/2)M_{r,r+\alpha}(\delta/2)\right] \tag{3.1}$$

b. For odd r and $j = 0, 1, \ldots, r - 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \frac{2}{\delta}\left[\sum_{k=0}^{[\frac{1}{2}(j-1)]} M_{r,2k+m}(p/2)N_{2r,\alpha}(\delta/2) - \sum_{k=0}^{\frac{1}{2}(r-3)+m} M_{r,2k+1-m}(p/2)N_{2r,r+\beta-1}(\delta/2) + \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(p-1)-m} M_{r,2k+m}(p/2)N_{2r,2r+\alpha}(\delta/2)\right] \tag{3.2}$$

where, in both (a) and (b) above and in Theorems 2 and 3 below,

$$\alpha = j - 2k - m, \quad \beta = j - 2k + m, \quad \text{and} \quad m = \begin{cases} 1, & j \text{ even} \\ 0, & j \text{ odd.} \end{cases}$$

Further, $[S]$ = the greatest integer $\leqslant S$ and δ as defined below.

Proof: By Sylvester's matrix interpolation formula (see [6]), we have

$$M_{r,j}(X) = \frac{1}{\lambda_1 - \lambda_2}\{[M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2)]X - [\lambda_1 M_{r,j}(\lambda_1) - \lambda_2 M_{r,j}(\lambda_2)]I\}, \tag{3.3}$$

where λ_1, λ_2 are distinct eigenvalues of X as defined in Lemma 1. It is easy to see that $\lambda_1 = (p + \delta)/2$, $\lambda_2 = (p - \delta)/2$, where $\delta = \sqrt{(p^2 - 4q)}$. Now

$$M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) = M_{r,j}\left(\frac{p+\delta}{2}\right) - M_{r,j}\left(\frac{p-\delta}{2}\right). \tag{3.4}$$

Using Lemma 2, (3.4) becomes

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= \sum_{k=0}^{j} M_{r,k}(p/2)[M_{r,j-k}(\delta/2) - M_{r,j-k}(-\delta/2)] \\ &\quad - \sum_{k=j+1}^{r-1} M_{r,k}(p/2)(M_{r,r+j-k}(\delta/2) - M_{r,r+j-k}(-\delta/2)). \end{aligned} \tag{3.5}$$

Let r and j both be even. Breaking the summation on the right side of (3.5) into even and odd values of k and then using (2.2), we obtain

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= 2\sum_{k=1,3,\ldots}^{j-1} M_{r,k}(p/2)M_{r,j-k}(\delta/2) \\ &\quad - 2\sum_{k=j+1,j+3,\ldots}^{r-1} M_{r,k}(p/2)M_{r,r+j-k}(\delta/2). \end{aligned}$$

Changing k to $2k + 1$, because k takes only odd values, we obtain

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= 2\sum_{k=0}^{\frac{1}{2}(j-2)} M_{r,2k+1}(p/2)M_{r,j-2k-1}(\delta/2) \\ &\quad - 2\sum_{k=j/2}^{\frac{1}{2}(r-2)} M_{r,2k+1}(p/2)M_{r,r+j-2k-1}(\delta/2). \end{aligned} \tag{3.6}$$

Now, by definition of $M_{r,j}(X)$ and Lemma 1, we have

$$M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(nr+j)!}[U_{nr+j}X - qU_{nr+j-1}I]. \tag{3.7}$$

Equating the coefficients of X in (3.7) and (3.3) and then making use of (3.6), we get (3.1) for even j. For odd j, (3.1) and (3.2) are similarly proved.

4. SUMMATION FORMULAS FOR $S_n(x)$

For Chebychev polynomials $S_n(x)$ of the first kind, we prove the following theorem. Let $x = \cos\theta$ and $y = \sin\theta$.

Theorem 2. a. Let r be such that $r/2$ is even, and $j = 0, 1, \ldots, r - 1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\left\{\sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,2k+m}(x)M_{r,\alpha}(y) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,2k+m}(x)M_{r,r+\alpha}(y)\right\}.$$

b. Let r be such that $r/2$ is odd, and $j = 0, 1, \ldots, r - 1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\left\{\sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,2k+m}(x) N_{r,\alpha}(y) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,2k+m}(x) N_{r,r+\alpha}(y)\right\}.$$

c. Let r be odd, $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\left\{\sum_{k=0}^{\frac{1}{2}(r-3)+m} (-1)^{\frac{1}{2}(r+\beta)} M_{r,2k+1-m}(x) M_{2r,r+\beta-1}(y) + \sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,2k+m}(x) M_{2r,\alpha}(y) + \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-1)-m} (-1)^{\frac{1}{2}(2r+\alpha-1)} M_{r,2k+m}(x) M_{2r,2r+\alpha}(y)\right\}.$$

Proof: If we write $x = \cos\theta$ and let $p = 2x$ and $q = 1$, then $U_n(p, q)$ are the Chebychev polynomials of the first kind, i.e.,

$$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \quad (n \geqslant 0),$$

where

$$S_{n+2} = 2xS_{n+1} - S_n, \quad \text{with } S_0 = 0 \text{ and } S_1 = 1.$$

We shall prove (a) and (b). Now

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!\,\sin\theta}$$

$$= \frac{1}{\sin\theta}\sum_{n=0}^{\infty} \frac{(-1)^n}{(nr+j)!}\left[\frac{e^{i(nr+j)\theta} - e^{-i(nr+j)\theta}}{2i}\right]$$

$$= \frac{1}{2i\sin\theta}\sum_{n=0}^{\infty} \frac{(-1)^n}{(nr+j)!}[(e^{i\theta})^{nr+j} - (e^{-i\theta})^{nr+j}]$$

$$= \frac{1}{2i\sin\theta}[M_{r,j}(e^{i\theta}) - M_{r,j}(e^{-i\theta})].$$

Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{2iy}[M_{r,j}(x+iy) - M_{r,j}(x-iy)]. \tag{4.1}$$

Now, by Lemma 2,

$$M_{r,j}(x+iy) - M_{r,j}(x-iy) = \sum_{k=0}^{j} M_{r,k}(x)[M_{r,j-k}(iy) - M_{r,j-k}(-iy)] - \sum_{k=j+1}^{r-1} M_{r,k}(x)[M_{r,r+j-k}(iy) - M_{r,r+j-k}(-iy)]. \tag{4.2}$$

First, let j be even. Breaking up the right-hand side of (4.2) into summations over even and odd values of k and making use of (2.2), we obtain

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) = &\sum_{k=1,3,\ldots}^{j-1} 2M_{r,k}(x)M_{r,j-k}(iy) \\ &- \sum_{k=j+1,j+3,\ldots}^{r-1} 2M_{r,k}(x)M_{r,r+j-k}(iy). \end{aligned} \tag{4.3}$$

Now, since r is even, $r/2$ is an integer that is either even or odd. First, let $r/2$ be *even*. By (2.7), (4.3) then becomes

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) = &2\sum_{k=1,3,\ldots}^{j-1} (i)^{j-k}M_{r,k}(x)M_{r,j-k}(y) \\ &- 2\sum_{k=j+1,j+3,\ldots}^{r-1} (i)^{r+j-k}M_{r,k}(x)M_{r,r+j-k}(y). \end{aligned} \tag{4.4}$$

If $r/2$ is *odd*, then again making use of (2.7), (4.3) becomes

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) = &2\sum_{k=1,3,\ldots}^{j-1} (i)^{j-k}M_{r,k}(x)N_{r,j-k}(y) \\ &- 2\sum_{k=j+1,j+3,\ldots}^{r-1} (i)^{r+j-k}M_{r,k}(x)N_{r,r+j-k}(y). \end{aligned} \tag{4.5}$$

Note that the power of i in all the summations in (4.4) and (4.5) is odd, so that when we substitute (4.4) and (4.5) in (4.1) and cancel i from the numerator and denominator, the remaining power of i will be an even integer. Then (4.1) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\Bigg[&\sum_{k=1,3,\ldots}^{j-1} (-1)^{\frac{1}{2}(j-k-1)}M_{r,k}(x)M_{r,j-k}(y) \\ &- \sum_{k=j+1,j+3,\ldots}^{r-1} (-1)^{\frac{1}{2}(r+j-k-1)}M_{r,k}(x)M_{r,r+j-k}(y)\Bigg] \end{aligned} \tag{4.6}$$

when $r/2$ is *even*, and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\Bigg[&\sum_{k=1,3,\ldots}^{j-1} (-1)^{\frac{1}{2}(j-k-1)}M_{r,k}(x)N_{r,j-k}(y) \\ &- \sum_{k=j+1,j+3,\ldots}^{r-1} (-1)^{\frac{1}{2}(r+j-k-1)}M_{r,k}(x)N_{r,r+j-k}(y)\Bigg] \end{aligned} \tag{4.7}$$

when $r/2$ is *odd*.

Replacing k by $2k+1$ in the right-hand side of (4.6) and (4.7), we finally get (a) and (b) for even j. By adopting similar techniques, we get (a) and (b) for odd j and (c).

5. SUMMATION FORMULAS FOR $T_n(x)$

Theorem 3. For the Chebychev polynomials $T_n(x)$ of the second kind, the following summation formulas hold.

a. Let r be such that $r/2$ is even and $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) M_{r,\,\beta-1}(y) - \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\beta-1)} M_{r,\,2k+1-m}(x) M_{r,\,r+\beta-1}(y).$$

b. Let r be such that $r/2$ is odd, $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) N_{r,\,\beta-1}(y) - \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\beta-1)} M_{r,\,2k+1-m}(x) N_{r,\,r+\beta-1}(y).$$

c. Let r be odd, $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) M_{2r,\,\beta-1}(y) - \sum_{k=0}^{\frac{1}{2}(r-1)-m} (-1)^{\frac{1}{2}(r+\alpha)} M_{r,\,2k+m}(x) M_{2r,\,r+\alpha}(y) + \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-3)+m} (-1)^{\frac{1}{2}(2r+\beta-1)} M_{r,\,2k+1-m}(x) M_{2r,\,2r+\beta-1}(y).$$

Proof: The proof follows the same technique as in Theorem 2 and is therefore omitted. Notice that the power of (-1) in each of the above summations is an integer.

Remark. Since

$$S_n(x) = \frac{\sin n\theta}{\sin\theta} \quad \text{and} \quad T_n(x) = \cos n\theta,$$

summation formulas in Theorems 2 and 3 also give those for

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \cos(nr+j)\theta}{(nr+j)!}.$$

For example, formula (a) in Theorem 2 can be expressed as

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!} = \sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,\,2k+m}(\cos\theta) M_{r,\,\alpha}(\sin\theta) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,\,2k+m}(\cos\theta) M_{r,\,r+\alpha}(\sin\theta).$$

REFERENCES

1. R. Barakat. "The Matrix Operator $e^{\underline{X}}$ and the Lucas Polynomials." *J. Math. Phys.* 43 (1964):332-35.
2. E. Lucas. *Théorie des nombres*. Paris: Albert Blanchard, 1961.
3. J. G. Mikusinski. "Sur les fonctions $k_n(x) = \sum_{\nu=0}^{\infty}(-1)^{\nu} x^{n+k\nu}/(n + k\nu)!$, ($k = 1, 2, \ldots; n = 0, 1, \ldots, k - 1$)." *Ann. Soc. Polon. Math.* 21 (1948): 46-51.
4. S. P. Pethe & A. Sharma. "Functions Analogous to Completely Convex Functions." *Rocky Mountain J. Math.* 3, no. 4 (1973):591-617.
5. S. Pethe. "On Lucas Polynomials and Some Summation Formulas for Chebychev Polynomial Sequences via Them." *The Fibonacci Quarterly* 22, no. 1 (1984): 61-69.
6. H. W. Turnbull & A. C. Aitken. *An Introduction to the Theory of Canonical Matrices*. New York: Dover Publications, 1961.
7. J. E. Walton. "Lucas Polynomials and Certain Circular Functions of Matrices." *The Fibonacci Quarterly* 14, no. 1 (1976):83-87.

◆◇◆◇◆