EULER'S PARTITION IDENTITY—ARE THERE ANY MORE LIKE IT?

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INTRODUCTION

A *partition* of a positive integer n is defined as a way of writing n as a sum of positive integers. Two such ways of writing n in which the parts merely differ in the order in which they are written are considered the same partition. We shall denote by $p(n)$ the number of partitions of n. Thus, for example, since 5 can be expressed by

$$5,\quad 4+1,\quad 3+2,\quad 3+1+1,\quad 2+2+1,\quad 2+1+1+1, \text{ and } 1+1+1+1+1,$$

we have $p(5) = 7$.

The function $p(n)$ is referred to as the number of *unrestricted partitions* of n to make clear that no restrictions are imposed upon the way in which n is partitioned into parts. In this paper, we shall concern ourselves with certain restricted partitions, that is, partitions in which some kind of restriction is imposed upon the parts. Specifically, we shall consider identities valid for all positive integers n of the general type

$$p'(n) = p''(n), \tag{1}$$

where $p'(n)$ is the number of partitions of n where the parts of n are subject to a first restriction and $p''(n)$ is the number of partitions of n where the parts of n are subject to an entirely different restriction.

The most celebrated identity of this type is due to Euler [4], who discovered it in 1748.

Theorem 1 (Euler)

The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Thus, for example, the partitions of 9 into distinct parts are

$$9,\quad 8+1,\quad 7+2,\quad 6+3,\quad 6+2+1,\quad 5+4,\quad 5+3+1,\quad 4+3+2,$$

that is, there are 8 such partitions, and the partitions of 9 into odd parts are

$$9,\quad 7+1+1,\quad 5+3+1,\quad 5+1+1+1+1,\quad 3+3+3,\quad 3+3+1+1+1,$$
$$3+1+\cdots+1,\quad 1+1+\cdots+1,$$

so that there are also 8 partitions of 9 into odd parts.

For a proof of this theorem by combinatorial methods, see [6], and by means of generating functions, see [2] or [3].

In [2], Alder has given a survey of the existence and nonexistence of generalizations of Euler's partition identity and, in [3], he has shown how to use generating functions to discover and prove the existence and the nonexistence of certain generalizations of this identity.

The use of generating functions, however, is by no means the only method for discovering partition identities or for proving their existence or nonexistence. Other methods, particularly those likely to produce positive results, that is, suggesting the existence of new partition identities, need, therefore, to be developed. Other points of view in looking at the possibility of the existence of such identities need to be encouraged. One such method is used in this paper. It is used to show that a certain generalization of a known partition identity cannot exist. It may well be, however, that as of yet unthought of techniques may prove successful in discovering a generalization.

In 1974 D. R. Hickerson [5] proved the following generalization of Euler's partition identity.

Theorem 2 (Hickerson)

If $f(r, n)$ denotes the number of partitions of n of the form $b_0 + b_1 + b_2 + \cdots + b_s$, where, for $0 \leqslant i \leqslant s - 1$, $b_i \geqslant rb_{i+1}$, and $g(r, n)$ denotes the number of partitions of n where each part is of the form $1 + r + r^2 + \cdots + r^i$ for some $i \geqslant 0$, then

$$f(r, n) = g(r, n). \tag{2}$$

Thus, for example, for $r = 2$, the partitions of 9 of the first type are

$9,\quad 8+1,\quad 7+2,\quad 6+3,\quad 6+2+1,$

so that $f(2, 9) = 5$, and the partitions of 9 of the second type, that is, where each part is chosen from the set $\{1, 3, 7, \ldots\}$, are

$7+1+1,\quad 3+3+3,\quad 3+3+1+1+1,\quad 3+1+\cdots+1,$ and $1+1+\cdots+1,$

so that also $g(2, 9) = 5$.

Hickerson gave a proof of this theorem, both by combinatorial methods and by means of generating functions.

In this paper we are addressing the question: Do there exist identities of the type given by Theorem 2, where the inequality $b_i \geqslant rb_{i+1}$ is replaced by $b_i > rb_{i+1}$?

THE NONEXISTENCE OF CERTAIN TYPES OF PARTITION IDENTITIES OF THE EULER TYPE

We shall consider the question stated above in the following more specific form: If $f(r, n)$ denotes the number of partitions of n of the form $b_0 + b_1 + \cdots + b_s$, where, for $0 \leqslant i \leqslant s - 1$, $b_i > rb_{i+1}$, and $g(r, n)$ denotes the number of partitions of n, where each part is taken from a set of integers S_r, for which r do there exist sets S_r such that $f(r, n) = g(r, n)$?

We know, of course, that for $r = 1$, there exists such a set, since Euler's partition theorem states that S_1 is the set of all positive odd integers. The question—whether there exist other values of r for which there exist sets S_r, so that (2) holds for all positive integers n—was posed at an undergraduate seminar on Number Theory by the first two authors in the Winter quarter 1983, and was answered with proof for all integers $r \geqslant 2$ by Jeffrey Lewis, namely as follows:

Theorem 3 (Lewis)

The number $f(r,n)$ of partitions of n of the form $b_0 + b_1 + \cdots + b_s$, where, for $0 \leqslant i \leqslant s - 1$, $b_i > rb_{i+1}$, r a positive integer, is not, for all n, equal to the number of partitions of n into parts taken from any set of integers whatsoever unless $r = 1$.

Proof of Theorem 3: We shall prove this theorem by contradiction. Let us assume that for some integer $r \geqslant 2$ there exists a set S_r of positive integers—denote the number of partitions of n into parts taken from that set by $g(r, n)$—for which $f(r, n) = g(r, n)$ for all n.

Since $f(r, 1) = 1$, we see that $1 \in S_r$ [otherwise, $g(r, 1) = 0$]. Since $f(r, 2) = 1$, it follows that $2 \notin S_r$ [otherwise, $g(r, 2) = 2$]. Since $f(r, 3) = f(r, 4) = \cdots = f(r, r+1) = 1$, we conclude that $3 \notin S_r$, $4 \notin S_r$, ..., $r + 1 \notin S_r$.

Now $f(r, r+2) = 2$, since the partitions of $r + 2$ for which $b_i > rb_{i+1}$ are $(r+2)$ and $(r+1)+1$. It follows that $r + 2 \in S_r$ [otherwise, $g(r, r+2) = 1$).

Thus, we have verified the entries in Table 1 up to $n = r+2$. We will now complete the construction of this table.

Table 1. Determination of the Elements of S_r for r an Integer $\geqslant 2$

n	$f(r, n)$	$g(r, n)$ if $n \notin S_r$	$g(r, n)$ if $n \in S_r$	Conclusion
1	1	0	1	$1 \in S$
2	1	1	2	$2 \notin S$
3	1	1	2	$3 \notin S$
⋮	⋮	⋮	⋮	⋮
$r + 1$	1	1	2	$r + 1 \notin S$
$r + 2$	2	1	2	$r + 2 \in S$
$r + 3$	2	2	3	$r + 3 \notin S$
⋮	⋮	⋮	⋮	⋮
$2r + 2$	2	2	3	$2r + 2 \notin S$
$2r + 3$	3	2	3	$2r + 3 \in S$
$2r + 4$	3	4	5	Contradiction

Next we determine the least value of n for which $f(r, n) = 3$. This occurs if n is of either of the forms

$$n = b_0 + b_1 + 1 \quad \text{with} \quad b_0 > rb_1 \quad \text{and} \quad b_1 > r$$

or

$$n = b_0 + 2 \quad \text{with} \quad b_0 > 2r.$$

The least n for which the first can occur is

$$n = (r^2 + r + 1) + (r + 1) + 1 = r^2 + 2r + 3.$$

The least n for which the second can occur is

$$n = (2r + 1) + 2 = 2r + 3.$$

Since $2r + 3 < r^2 + 2r + 3$ for all positive r, it follows that $n = 2r + 3$ is the least value of n for which $f(r, n) = 3$. In that case, the partitions of $2r + 3$ for which $b_i > rb_{i+1}$ are $(2r+3)$, $(2r+2) + 1$, $(2r+1) + 2$. Now, since thus far only $1 \in S_r$ and $r + 2 \in S_r$, it follows that there are only two partitions of $2r + 3$ into parts taken from that set, namely

$$(r + 2) + 1 + \cdots + 1 \quad \text{and} \quad 1 + 1 + \cdots + 1,$$

so that we need $2r + 3 \in S_r$ in order to make $g(r, 2r+3) = 3$.

Now $f(r, 2r+4) = 3$, since the only partitions of $2r + 4$, with $b_i > rb_{i+1}$, are $(2r+4)$, $(2r+3) + 1$, $(2r+2) + 2$. (Note that it is here where we are using the fact that $r \geqslant 2$.) On the other hand, the partitions of $2r + 4$ into parts taken from the set $\{1, r+2, 2r+3\}$ are

$$(2r+3) + 1, \quad (r+2) + (r+2), \quad (r+2) + 1 + 1 + \cdots + 1, \quad 1 + 1 + \cdots + 1,$$

so that $g(r, 2r+3) = 4$ if $2r + 4 \notin S_r$ and $g(r, 2r+3) = 5$ if $2r + 4 \in S_r$, which is a contradiction.

The question arises whether Theorem 3 is true also for all values of $r > 1$. We have some partial answers to this question.

Theorem 4

The nonexistence of sets S_r given in Theorem 3 also applies to all r in any of the intervals $N \leqslant r < N + 1/2$, where N is any integer $\geqslant 2$.

Proof of Theorem 4: This proof is identical to that for Theorem 3, except that, in the construction of Table 1, the entries in the columns for n and the conclusions have to be changed by replacing r in every case by $[r]$, the grestest integer in r. Note that the condition that $r < N + 1/2$ is needed in the determination of the partitions of $2[r] + 3$ with $b_i > rb_{i+1}$, which are

$$(2[r] + 3), \quad (2[r] + 2) + 2, \text{ and } (2[r] + 1) + 2,$$

the latter satisfying the inequality, since

$$2[r] + 1 = 2N + 1 = 2\left(N + \frac{1}{2}\right) > 2r.$$

Now, for values of r for which $N + (1/2) \leqslant r < N + 1$, we have a method for proving the nonexistence of S_r for certain intervals, but have no method which will give a conclusion valid for all such intervals. We illustrate this method for intervals in the range $2.50 \leqslant r < 3.00$.

First we use the same method used in the construction of Table 1 to determine the elements of S_r for $r = 2.50$. (See Table 2.)

Since a contradiction is obtained for $n = 20$, it follows that for $r = 2.50$ no set S_r can exist for which $f(r, n) = g(r, n)$ for all positive integers n.

Next we note that Table 2 applies for all r with $2.50 \leqslant r < x/y$, where x/y is the least rational number > 2.50 for which both x and y appear as parts in a partition counted by $f(r, n)$ in Table 2; that is, we need to find the least rational number $x/y > 2.50$ for which $x + y \leqslant 20$. This clearly is $13/5$, since $13 + 5$ is a partition of 18 and, therefore, Table 2 would not be applicable for $r = 13/5$ because the partition of $18 = 13 + 5$ would not satisfy $13 > 5r$ for $r = 13/5$.

Thus, Table 2 is applicable for all r with $2.50 \leqslant r < 13/5$, and the nonexistence of the sets S_r follows for all r in this interval.

Table 2. Determination of the Elements of S_r for r = 2.50

n	$f(r, n)$	$g(r, n)$ if $n \notin S$	$g(n, r)$ if $n \in S$	Conclusion
1	1	0	1	$1 \in S$
2	1	1	2	$2 \notin S$
3	1	1	2	$3 \notin S$
4	2	1	2	$4 \in S$
5	2	2	3	$5 \notin S$
6	2	2	3	$6 \notin S$
7	2	2	3	$7 \notin S$
8	3	3	4	$8 \notin S$
9	3	3	4	$9 \notin S$
10	3	3	4	$10 \notin S$
11	4	3	4	$11 \in S$
12	5	5	6	$12 \notin S$
13	5	5	6	$13 \notin S$
14	5	5	6	$14 \notin S$
15	6	6	7	$15 \notin S$
16	7	7	8	$16 \notin S$
17	7	7	8	$17 \notin S$
18	8	7	8	$18 \in S$
19	9	9	10	$19 \notin S$
20	9	10	11	Contradiction

We now construct, by programming on a computer, a table similar to Table 2 for r = 13/5 (not shown here), obtaining a contradiction for n = 52. Next, we note that this table applies to all r with $13/5 \leqslant r < x/y$, where x/y is the least rational number > 13/5 for which $x + y \leqslant 52$. This clearly is 34/13, so that this table is applicable for all r with $13/5 \leqslant r < 34/13$. Constructing a table similar to Table 2 for r = 34/13, we obtain a contradiction for n = 136 and find that this table is valid for all r with $34/13 \leqslant r < 89/34$. Then, constructing the appropriate table for r = 89/34, we were unable to obtain a contradiction on the computer in the time available, that is, for $n \leqslant 181$.

Though we were unable to obtain a contradiction for r = 89/34 = 2.6176..., we were able to obtain one for r = 2.62, namely for n = 90 and, using the previously described method, to determine that this table is valid for all r with $2.62 \leqslant r < 21/8$. Then, considering $r \geqslant 21/8$, we were able to obtain contradictions for all $r < 32/11 = 2.909...$ for the values of n indicated in Table 3.

For values of $r \geqslant 32/11$, the corresponding tables again became so long that the time available on the computer to arrive at a contradiction was exceeded; thus, we have no conclusions for $32/11 \leqslant r < 3$.

For values of r between 1 and 2, the smallest value of r we considered was r = 1.08, for which we obtained a contradiction for n = 54. Using the same method as used for values of r in the interval $2.50 \leqslant r < 32/11$, it was possible to prove the nonexistence of S_r for all r in the short interval $1.08 \leqslant r < 25/23 = 1.0869...$.

To obtain results valid for larger intervals, we started with r = 1.25 and proved the nonexistence of S_r for all r in the interval $1.25 \leqslant r < 23/12 = 1.9166...$, as indicated in Table 4.

Table 3. The Nonexistence of S_r for $2.50 \leqslant r < 89/34 = 2.6176\ldots$ and $2.62 \leqslant r < 32/11 = 2.909\ldots$

Interval	Value of n for Which Contradiction Occurs
$2.50 \leqslant r < 13/5$	20
$13/5 \leqslant r < 34/13$	52
$34/13 \leqslant r < 89/34$	136
$89/34 \leqslant r < 55/21$	No conclusion
$55/21 \leqslant r < 21/8$	90
$21/8 \leqslant r < 8/3$	38
$8/3 \leqslant r < 11/4$	17
$11/4 \leqslant r < 14/5$	21
$14/5 \leqslant r < 17/6$	26
$17/6 \leqslant r < 20/7$	30
$20/7 \leqslant r < 23/8$	34
$23/8 \leqslant r < 26/9$	48
$26/9 \leqslant r < 29/10$	44
$29/10 \leqslant r < 32/11$	48

Table 4. The Nonexistence of S_r for $1.25 \leqslant r < 23/12 = 1.9166\ldots$

Interval	Value of n for Which Contradiction Occurs
$1.25 \leqslant r < 9/7$	18
$9/7 \leqslant r < 4/3$	18
$4/3 \leqslant r < 7/5$	14
$7/5 \leqslant r < 3/2$	14
$3/2 \leqslant r < 5/3$	10
$5/3 \leqslant r < 7/4$	18
$7/4 \leqslant r < 9/5$	18
$9/5 \leqslant r < 11/6$	21
$11/6 \leqslant r < 13/7$	24
$13/7 \leqslant r < 15/8$	28
$15/8 \leqslant r < 17/9$	33
$17/9 \leqslant r < 19/10$	36
$19/10 \leqslant r < 21/11$	39
$21/11 \leqslant r < 23/12$	42

For values of $r < 1.25$, as indicated above, the intervals for which a table similar to Table 2 is valid become very small. Considering the values of $r = 1.08, 1.09, \ldots, 1.20$, separately, we obtained a contradiction for each of them. For values of r close to 1, the time available on the computer to arrive at a contradiction was exceeded. This is not surprising, because we know that, for $r = 1$, we have the Euler identity and, therefore, no contradiction can be obtained. For values of r in the interval $1 < r < 1.25$, except for those listed above and for those in the interval $23/12 \leqslant r < 2$, we have no conclusions.

It is an interesting question whether Theorem 3 can be proved by a method valid for all nonintegral values of $r > 1$.

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