A NOTE ON INFINITE EXPONENTIALS

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(Submitted August 1983)

For $a > 0$, the sequence

$$a,\ a^a,\ a^{(a^a)},\ \ldots, \tag{1}$$

is convergent if and only if $a \in I = [e^{-e}, e^{1/e}]$. This result, which was known to Euler [5], and which has been rediscovered frequently, is capable of generalization in various directions (see [6] for a wide-ranging survey). For instance, Barrow [2] showed that if $a_n \in I$, $n = 1, 2, \ldots$, then the sequence

$$a_1,\ a_1^{a_2},\ a_1^{(a_2^{a_3})},\ \ldots, \tag{2}$$

is convergent also.

More recently [1], we have observed that if a is a complex number and if

$$a^z = \exp[z \log a], \quad (z \in \mathbb{C}),$$

where the principal value of the logarithm is taken, then the sequence (1) converges if a lies in

$$R = \{e^{te^{-t}} : |t| < 1\}.$$

On the boundary of R however, and in its exterior, both convergence and divergence may occur.

The sequence (2) was shown by Thron [7] to be convergent if $|\log a_n| \leqslant 1/e$, $n = 1, 2, \ldots$, but we do not know whether this holds in general if $a_n \in R$, $n = 1, 2, \ldots$.

The aim of the present note is to give a complete discussion of the behaviour of real sequences of the form

$$a,\ a^b,\ a^{(b^a)},\ a^{(b^{(a^b)})},\ \ldots, \quad (a, b > 0). \tag{3}$$

Such a sequence is of course a special case of (2), and so Barrow's result guarantees convergence for $(a, b) \in I \times I$, though the full region of convergence is actually much larger. The same problem was discussed and partially solved by Creutz and Sternheimer [4], who also presented considerable computational evidence concerning the region of convergence.

With $a, b > 0$, we let $\phi(x) = a^{b^x} (= a^{(b^x)})$, $-\infty < x < \infty$, and

$$\phi^{n+1}(x) = \phi o \phi^n(x) = \phi^n o \phi(x), \quad (n = 1, 2, \ldots).$$

The sequence (3) under consideration is then of the form

$$\phi(0),\ \phi(1),\ \phi^2(0),\ \phi^2(1),\ \ldots .$$

Theorem

The following statements are equivalent:

(i) Both

$\lim_{n\to\infty} \phi^n(0)$ and $\lim_{n\to\infty} \phi^n(1)$

exist finitely and are equal.

(ii) The function ϕ has precisely one fixed point c such that $|\phi'(c)| \leqslant 1$.

(iii) We can write

$$\log a = se^{-t} \quad \text{and} \quad \log b = te^{-s}, \quad (|st| \leqslant 1), \tag{4}$$

in a unique way.

The set of points ($\log a$, $\log b$) defined by statement (iii) is shaded in Figure 1 for the reader's convenience. We shall discuss it in more detail once the theorem is proved. Notice that $[-e, 1/e] \times [-e, 1/e]$ lies in the shaded set, as is implied by Barrow's result.

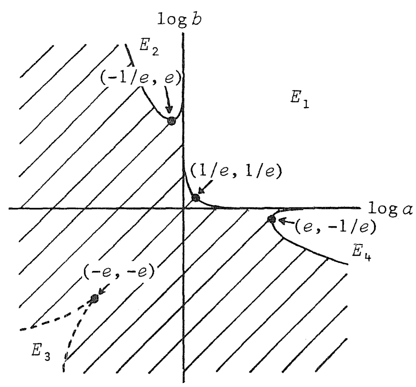

Figure 1

The behavior of the sequence at the remaining points, which will become clear in the course of the proof, is indicated below:

when ($\log a$, $\log b$) $\in E_1$, we have

$\lim_{n\to\infty} \phi_n(0) = \lim_{n\to\infty} \phi_n(1) = \infty$;

when ($\log a$, $\log b$) $\in E_2 \cup E_4$, we have

$\lim_{n\to\infty} \phi^{2n}(0) = \lim_{n\to\infty} \phi^{2n+1}(1) \neq \lim_{n\to\infty} \phi^{2n+1}(0) = \lim_{n\to\infty} \phi^{2n}(1) < \infty$,

when ($\log a$, $\log b$) $\in E_3$, we have

$\lim_{n\to\infty} \phi_n(0) < \lim_{n\to\infty} \phi_n(1) < 1$.

The equivalence of statements (ii) and (iii) is a special case of the following lemma.

Lemma 1

There is a one-to-one correspondence between the fixed points $c = e^s$ of ϕ, such that $|\phi'(c)| \leqslant 1$, and the representations of $(\log a, \log b)$ in the form (4).

Proof: To prove the lemma, note that

$$c = a^{b^c} = \exp(\exp(c \log b)\log a)$$

if and only if $c = e^s$, where

$$s = \exp(c \log b)\log a,$$

and s is of this form if and only if we can write

$$\log a = se^{-t} \text{ and } \log b = te^{-s}.$$

Since we then have

$$\phi'(c) = a^{b^c}b^c \log a \log b = c \exp(c \log b)\log a \log b = st,$$

the proof of the lemma is complete.

We now show that statements (i) and (ii) are equivalent. First we assume that $a, b > 1$ so that ϕ is increasing. Since

$$\phi''(x) = a^{b^x}b^x \log a(\log b)^2(1 + b^x \log a)$$

the function ϕ has no points of inflection and so has at most two fixed points. It is clear that

$$\phi^n(0) < \phi^n(1) < \phi^{n+1}(0), \quad (n = 1, 2, \ldots),$$

and so convergence occurs if and only if ϕ has at least one fixed point, in which case ϕ has exactly one fixed point c such that $|\phi'(c)| \leqslant 1$.

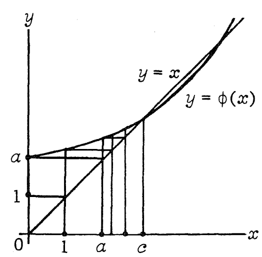

Figure 2

If ϕ has no fixed points, then we clearly have

$$\phi^n(0) \to \infty \quad \text{and} \quad \phi^n(1) \to \infty, \quad (n \to \infty).$$

Next we assume that $0 < a, b < 1$. Once again ϕ is increasing, but now it has one point of inflection, and so there may be one, two, or three fixed points. For $n = 1, 2, \ldots$, we have

$$\phi^n(0) < \phi^{n+1}(0) < \phi^{n+1}(1) < \phi^n(1),$$

and

$$\lim_{n\to\infty} \phi^n(0) = \lim_{n\to\infty} \phi^n(1) \tag{5}$$

is true if and only if ϕ has exactly one fixed point c. The condition $|\phi'(c)| \leqslant 1$ will then automatically be satisfied.

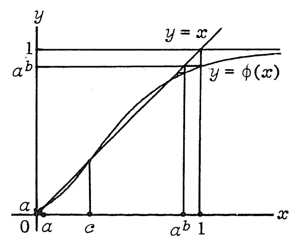

Figure 3

With more than one fixed point, as in Figure 3, both the limits in (5) exist, but they are not equal. This is an example of what Creutz and Sternheimer call "dual convergence."

Since the sequence a, a^b, $a^{(b^a)}$, ... is convergent if and only if the sequence b, b^a, $b^{(a^b)}$, ... is convergent, the cases $0<a<1<b$ and $0<b<1<a$ are equivalent. We may assume then, finally, that $0<a<1<b$. In this case, ϕ is decreasing and has a unique fixed point c.

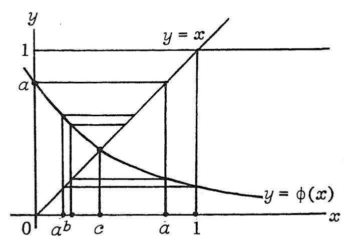

Figure 4

There are now four monotonic subsequences of interest. Indeed, for $n = 1, 2, \ldots$, we have

$$\phi^{2n-1}(1) < \phi^{2n}(0) < \phi^{2n+1}(1) < c < \phi^{2n+1}(0) < \phi^{2n}(1) < \phi^{2n-1}(0), \tag{6}$$

which is easily verified by induction, since $\phi^2 = \phi o \phi$ is increasing and has a fixed point at $x = c$. If $|\phi'(c)| > 1$, then no sequence of the form $\phi^n(x_0)$, $n = 1, 2, \ldots$, $x_0 > 1$, can converge to c, and so in this case we have another (but slightly different) example of dual convergence.

To prove that convergence does occur when $|\phi'(c)| \leqslant 1$, it is enough to show that ϕ^2 has in this case only one fixed point, namely c, since this would imply that

$$\lim_{n\to\infty} \phi^{2n}(0) = c = \lim_{n\to\infty} \phi^{2n}(1).$$

We are, therefore, reduced to proving that, for $0 < x < c$,

$$\phi(x) < \phi^{-1}(x),$$

or

$$a^{b^x} < (\log(\log x/\log a))/\log b.$$

With $\log a = se^{-t}$, $\log b = te^{-s}$, $c = e^s$, and $x = (1 - u)e^s$ (cf. the proof of Lemma 1), this becomes, for $0 < u < 1$,

$$\exp[-s(1 - e^{-ut})] < 1 + \frac{1}{t}\log\left(1 + \frac{1}{s}\log(1 - u)\right), \tag{7}$$

which we must show to be true when $s < 0$, $t > 0$, and $|st| = |\phi'(c)| \leqslant 1$. In fact, it is enough to prove (7) when $s = -1/t$, which we now do.

Lemma 2

For $t > 0$ and $0 < u < 1$, we have

$$\exp\left[\frac{1 - e^{-ut}}{t}\right] < 1 + \frac{1}{t}\log\left(1 + t\log\frac{1}{1 - u}\right).$$

Proof: To prove the lemma, note that

$$\exp\left[\frac{1 - e^{-ut}}{t}\right] < \exp\left[\frac{1}{t}\log(1 + ut)\right] = (1 + ut)^{1/t}, \quad (t > 0,\ u > 0),$$

and so, since there is equality at $u = 0$, it is enough to show that

$$\frac{d}{du}[(1 + ut)^{1/t}] < \frac{d}{du}\left[1 + \frac{1}{t}\log\left(1 + t\log\frac{1}{1 - u}\right)\right], \quad (t > 0,\ 0 < u < 1),$$

which is equivalent to

$$1 + t\log\frac{1}{1 - u} < \frac{(1 + ut)^{1 - 1/t}}{1 - u}, \quad (t > 0,\ 0 < u < 1).$$

Again there is equality at $u = 0$, and so it is enough to show that

$$\frac{d}{du}\left[1 + t\log\frac{1}{1 - u}\right] < \frac{d}{du}\left[\frac{(1 + ut)^{1 - 1/t}}{1 - u}\right], \quad (t > 0,\ 0 < u < 1),$$

which is equivalent to

$$(1 + ut)^{1/t} < \frac{t - 1}{t} + \frac{1 + ut}{t(1 - u)} = \frac{1}{1 - u} + \frac{1}{t}\left(\frac{1}{1 - u} - 1\right), \tag{8}$$

$$(t > 0,\ 0 < u < 1).$$

However, since

$$(1 + ut)^{1/t} < e^u < \frac{1}{1 - u}, \quad (t > 0,\ 0 < u < 1),$$

the estimate (8) does in fact hold. This completes the proof of Lemma 2 and also that of our theorem.

We now discuss the mapping $x = se^{-t}$, $y = te^{-s}$, $|st| \leqslant x$, which gives rise to the region in Figure 1. First, it is clear that, for $k = 1, 2, 3, 4$, the k^{th} quadrant in the st-plane is mapped into the k^{th} quadrant of the xy-plane. Next we observe that the mapping is one-to-one for $t > 0$ and $|st| \leqslant 1$. This follows from Lemma 1, if we recall from the proof of the theorem that, for $b > 1$, the function ϕ has a fixed point c with $|\phi'(c)| \leqslant 1$ if and only if this fixed point is unique. By symmetry, the mapping is also one-to-one for $s > 0$.

It is easy to check with a little calculus that the boundary of the image of $\{(s, t) : |st| \leqslant 1\}$ takes the form shown in Figure 1 in the first, second, and fourth quadrants. In the third quadrant, however, the mapping is not one-to-one and a more detailed discussion is required.

If $st = 1$ $(s, t < 0)$ and $x = se^{-t}$, $y = te^{-s}$, then

$$y < x < -e, \quad (s < -1) \quad \text{and} \quad x < y < -e, \quad (s > -1). \tag{9}$$

For instance, if $s < -1$, then the inequality

$$x = se^{-1/s} > s^{-1}e^{-s} = y$$

is equivalent (on putting $\sigma = -s$) to

$$2 \log \sigma < \sigma - 1/\sigma, \quad (\sigma > 1),$$

which is easily verified by differentiation. The maximum value of $se^{-1/s}$ for $s < 0$ occurs when $s = -1$, and so, for $x < -e$, the equation $x = se^{-1/s}$ has two solutions s_1, s_2 with $s_2 < -1 < s_1 < 0$. If $s_1t_1 = 1 = s_2t_2$ and

$$y_1(x) = t_1e^{-s_1}, \; y_2(x) = t_2e^{-s_2}, \quad (x < -e),$$

then y_1, y_2 are smooth functions in $(-\infty, -e)$ and, by (9),

$$y_2(x) < x < y_1(x) < -e, \quad (x < -e).$$

It is easy to check that

$$\lim_{x \to -e} y_1(x) = -e = \lim_{x \to -e} y_2(x)$$

and, using the chain rule, that

$$\lim_{x \to -e} y_1'(x) = 1 = \lim_{x \to -e} y_2'(x).$$

Hence, the image of $st = 1$ has a cusp at $(-e, -e)$.

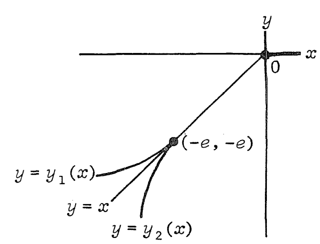

Figure 5

We now claim that the set

$$\{(x, y) : x < -e, \; y_2(x) \leqslant y \leqslant y_1(x)\}$$

is covered twice by the mapping and that the remainder of the third quadrant is covered once. These facts could be verified directly, or we can deduce them from Lemma 1 as follows.

Since, for $0 < a$ and $b < 1$, the maximum value of $\phi'(x)$ is $(-\log b)/e$ (this occurs when $1 + b^x \log a = 0$), we see that ϕ has exactly one fixed point c (and $\phi'(c) \leqslant 1$) if $0 < a < 1$ and $e^{-e} \leqslant b < 1$. This means, by Lemma 1, that

$$\{(x, y) : -\infty < x < 0, -e \leqslant y < 0\}$$

is covered exactly once.

If $0 < b < e^{-e}$, however, then there are numbers a_1 and a_2 with $0 < a_1 < a_2 < e^{-e}$ such that

$$y_1(\log a_1) = \log b = y_2(\log a_2),$$

and then the corresponding functions

$$\phi_1(x) = a_1^{b^x} \quad \text{and} \quad \phi_2(x) = a_2^{b^x}$$

each has a fixed point with derivative 1 (see the proof of Lemma 1). Since $\phi(x) = a^{b^x}$ is strictly monotonic in a when b, x are fixed, we see that for all $a \in [a_1, a_2]$, the function ϕ has two fixed points c such that $\phi'(c) \leqslant 1$. If $a \notin [a_1\ a_2]$, however, the function ϕ has only one such fixed point. By Lemma 1, this establishes the claim.

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