A NOTE ON BINOMIAL COEFFICIENTS AND CHEBYSHEV POLYNOMIALS

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In [1] the author gave a demonstration in a slightly different notation of the following property of binomial coefficients: for every integer n, and $k < n$,

$$\sum_{j=0}^{k} 2^{-(n-k+j-1)}\binom{n-k+j-1}{j} + \sum_{j=0}^{n-k-1} 2^{-(k+j)}\binom{k+j}{j} = 2. \tag{1}$$

In this note we shall be concerned with an application of (1) to a problem involving Chebyshev polynomials of the first kind. Remember that the Chebyshev polynomial of the first kind $T_n(x)$ is defined in $-1 \leqslant x \leqslant 1$ as

$$T_n(x) = \cos n(\arccos x).$$

For the sake of convenience we sometimes use the notation T_n instead of $T_n(x)$. We shall use the following two identities (see, e.g., [2] for the proofs):

(a) for every integer n,

$$T_{-n} = T_n; \tag{2}$$

(b) for r, $n > 0$,

$$x^r T_n(x) = 2^{-r} \sum_{i=0}^{r} \binom{r}{i} T_{n-r+2i}(x). \tag{3}$$

Proposition 1

For every integer $n \geqslant 1$, we have

$$\sum_{i=0}^{n-1} x^i T_{n-i-1} = \sum_{i=0}^{n-1} T_{n-2i-1}. \tag{4}$$

Proof: Using (3), we can write the summation on the left as

$$S = \sum_{i=0}^{n-1} 2^{-i} \sum_{j=0}^{i} \binom{i}{j} T_{n-2(i-j)-1}$$

or, changing indexes,

$$S = \sum_{i=0}^{n-1} \sum_{j=0}^{i} 2^{-(n-i+j-1)} \binom{n-i+j-1}{j} T_{-n+2i+1}. \tag{5}$$

We denote by C_i the term involving $T_{-n+2i+1}$ in (5), $i = 0, \ldots, n-1$. Then

$$S = C_0 + C_1 + C_2 + \cdots + C_{n-1} \tag{6}$$

and also, in reverse order,

$$S = C_{n-1} + C_{n-2} + \cdots + C_1 + C_0. \tag{7}$$

Adding (6) and (7) term by term, we can write

$$2S = (C_0 + C_{n-1}) + (C_1 + C_{n-2}) + \cdots + (C_{n-1} + C_0)$$
$$= \sum_{k=0}^{n-1} (C_k + C_{n-k-1}). \tag{8}$$

Now, from (5),

$$C_k = \sum_{j=0}^{k} 2^{-(n-k+j-1)} \binom{n-k+j-1}{j} T_{-n+2k+1}$$
$$C_{n-k-1} = \sum_{j=0}^{n-k-1} 2^{-(k+j)} \binom{k+j}{j} T_{n-2k-1}.$$

Because of (2), $T_{n-2k-1} = T_{-n+2k+1}$. Therefore,

$$C_k + C_{n-k-1} = \left[\sum_{j=0}^{k} 2^{-(n-k+j-1)} \binom{n-k+j-1}{1} + \sum_{j=0}^{n-k-1} 2^{-(k+j)} \binom{k+j}{j}\right] T_{n-2k-1}.$$

But the coefficient of T_{n-2k-1} in the above formula is just the expression (1), which is always equal to 2, regardless of the values of n and k. Hence, we can rewrite (8) as

$$2S = \sum_{k=0}^{n-1} 2T_{n-2k-1},$$

and the proposition is thus proved.

Corollary 1

For every integer $n \geqslant 1$,

$$T'_n = n\sum_{i=0}^{n-1} T_{n-2i-1} = n\sum_{i=0}^{n-1} x^i T_{n-i-1}, \tag{9}$$

where T'_n is the first derivative of T_n with respect to x.

Proof: Let $f = \arccos x$ and $\omega = e^{if} = \cos f + i \sin f$. Then, from the definition of $T_n(x)$,

$$\frac{T'_n(x)}{n} = \frac{\sin nf}{\sin f} = \frac{\omega^n - \omega^{-n}}{\omega - \omega^{-1}} = \frac{\omega^{n-1}(1 - \omega^{-2n})}{1 - \omega^{-2}} = \sum_{i=0}^{n-1} \omega^{n-2i-1}$$
$$= \frac{1}{2}\sum_{i=0}^{n-1} (\omega^{n-2i-1} + \omega^{-n+2i+1}) = \sum_{i=0}^{n-1} \cos(n - 2i - 1)f = \sum_{i=0}^{n-1} T_{n-2i-1},$$

and from Proposition 1, the conclusion follows.

A proof of the first equality of (9) is found also in [3].

Remark: Remember that the Chebyshev polynomial of the second kind $U_n(x)$ is defined as

$$U_n(f) = \frac{\sin(n+1)f}{\sin f} \quad \text{(notation as in the proof of Corollary 1).}$$

From Corollary 1 and the known result $T_n' = nU_{n-1}$, it follows that

$$U_n = \sum_{i=0}^{n} T_{n-2i}.$$

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