SOME GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

In classical usage the *fundamental* and *primordial* second-order recurrences are those of Fibonacci and Lucas, $\{F_n\}$ and $\{L_n\}$, defined by the linear homogeneous recurrence relation

$$U_n = U_{n-1} + U_{n-2}, \quad n > 2, \tag{1.1}$$

with initial conditions $F_1 = 1$, $F_2 = 1$, and $L_1 = 1$, $L_2 = 3$. They are usually generalized by altering the recurrence relation or the initial conditions as described by Horadam [4].

There have been many generalizations of the Fibonacci numbers (cf. Bergum & Hoggatt [1] and Shannon [11]), but fewer published attempts to generalize the corresponding Lucas numbers, though those of Hoggatt and Bicknell-Johnson (cf. [4]) are notable exceptions.

We believe that the following exposition is a useful addition to the literature because, unlike other papers, which concentrate on particular properties, we focus on the unexpected structure of the generalized recurrence relation. This complements the existing literature because the solution of our recurrence relation is the one used by the authors to develop various properties of these sequences. The corresponding approach for the Fibonacci numbers has been applied by Hock and McQuiston [3]. From the simple form of the recurrence relation as revealed here, we specify some particular generalized sequences and two special properties that will be of use to future researchers of the abritrary-order recurrences who utilize the coefficients of the recurrence relation.

We choose here to generalize the Lucas sequence by considering the r^{th}-(arbitrary)-order linear recurrence relation

$$v_n^{(r)} = v_{n-r+1}^{(r)} + v_{n-r}^{(r)}, \quad n \geqslant r > 1, \tag{1.2}$$

and initial conditions $v_n^{(r)} = 0$ if $0 < n < r - 1$, $v_{r-1}^{(r)} = r - 1$ and $v_0^{(r)} = r$. The notation is due to Williams [12] and has been used since then by several authors in studying r^{th}-order recurrences.

Thus, $\{v_n^{(2)}\} \equiv \{L_n\}$, and the accompanying table displays the first 16 terms of $\{v_n^{(r)}\}$ for $r = 2, 3, 4, 5, 6$.

Table 1. Generalized Lucas Numbers for $n \geqslant 0$

r \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364	2207
3	3	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68	90
4	4	0	0	3	4	0	3	7	4	3	10	11	7	13	21	18	20
5	5	0	0	0	4	5	0	0	4	9	5	0	4	13	14	5	4
6	6	0	0	0	0	5	6	0	0	0	5	11	6	0	0	5	16

For example, from the table, we have that

$$v_n^{(4)} = v_{n-3}^{(4)} + v_{n-4}^{(4)} \quad \text{or} \quad 20 = v_{16}^{(4)} = v_{13}^{(4)} + v_{12}^{(4)} = 13 + 7.$$

We propose to consider some of the properties of $\{v_n^{(r)}\}$ that arise from the interesting fact that all but three of the coefficients in the recurrence relation are zero.

2. GENERAL TERMS

The auxiliary equation associated with the recurrence relation (1.2) is

$$x^r - x - 1 = 0, \tag{2.1}$$

which we assume has distinct roots, α_j, $j = 1, 2, \ldots, r$. In fact, $v_n^{(r)}$ is (in the terminology of Macmahon [8]) the homogeneous product sum of weight n of the quantities α_j. It is the sum of a number of symmetric functions formed from a partition of n as elaborated in Shannon [10]. The first three cases are

$$\begin{aligned} v_1^{(r)} &= P_{r1} &&= \sum \alpha_j \\ v_2^{(r)} &= P_{r1}^2 + P_{r2} &&= \sum \alpha_j^2 + \sum \alpha_i \alpha_j \\ v_3^{(r)} &= P_{r1}^3 + 2P_{r1}P_{r2} + P_{r3} &&= \sum \alpha_j^3 + \sum \alpha_i^2 \alpha_j + \sum \alpha_i \alpha_j \alpha_k \end{aligned}$$

in which P_{rm} is $(-1)^{m+1}$ times the sum of the α_j taken m at a time as in the theory of equations. More generally,

$$v_n^{(r)} = \sum_{\Sigma\lambda = n} \prod_{i=1}^{r} \alpha_i^{\lambda_i},$$

so that since $P_{rm} = 0$ except for P_{rr} and $P_{r,r-1}$, which are unity, we have

$$v_n^{(r)} = \sum_{j=1}^{r} \alpha_j^n \quad \text{for } n = 1, 2, \ldots, r. \tag{2.2}$$

Then, if we assume the result (2.2) is true for $n = k - 1$:

$$\begin{aligned} v_k^{(r)} &= v_{k-r+1}^{(r)} + v_{k-r}^{(r)} \\ &= \sum_{j=1}^{r} (\alpha_j^{k-r+1} + \alpha_j^{k-r}) = \sum_{j=1}^{r} \alpha_j^{k-r}(\alpha_j + 1) = \sum_{j=1}^{r} \alpha_j^{k-r} \alpha_j^r = \sum_{j=1}^{r} \alpha_j^k. \end{aligned}$$

By the Principle of Mathematical Induction, we get

$$v_n^{(r)} = \sum_{j=1}^{r} \alpha_j^n. \tag{2.3}$$

For example,

$$v_n^{(2)} = (1.61803)^n + (-0.61803)^n,$$

the well-known result for the Lucas numbers.

Similarly, for instance, with $i^2 = -1$,

$$v^{(3)} = (1.32472)^n + (-0.66236 + 0.07165i)^n + (-0.66236 - 0.07165i)^n,$$

and

$$v^{(4)} = (1.22075)^n + (-0.7245)^n + (-0.2481 + 1.341i)^n + (-0.2481 - 1.0341i)^n.$$

3. GENERAL PROPERTIES

Among the various properties that can be investigated, we focus on two that follow directly from (2.1) and (1.2).

For odd values of r, (2.1) has the real solution

$$\alpha = (1 + \alpha)^{1/r}$$

which leads to the approximation

$$\alpha \doteqdot r/(r - 1); \tag{3.1}$$

for even values of r, we get

$$\alpha \doteqdot \pm(1 + \alpha/r)$$

or

$$\alpha \doteqdot r/(r \pm 1). \tag{3.2}$$

These are the initial approximate values which, by repeated iterations, converge to the real roots. Furthermore, we observe in (3.1) and (3.2) that as r increases, α approaches unity, which can be confirmed readily with a few numerical examples.

For notational convenience, we assume that u_n exists for $n < 0$. Then, for any $j \in \mathbf{Z}_+$,

$$v_n^{(r)} = \sum_{i=0}^{j} \binom{j}{i} v_{n-rj+1}^{(r)}. \tag{3.3}$$

Proof: We use induction on j. When $j = 1$, (3.3) reduces to the recurrence relation (1.2). Suppose the result is true for $j = 2, 3, \ldots, k - 1$. Then

$$\begin{aligned}\sum_{i=0}^{k} \binom{k}{i} v_{n-rk+i}^{(r)} &= \sum_{i=0}^{k-1} \binom{k-1}{i} v_{n-rk+i}^{(r)} + \sum_{i=1}^{k-1} \binom{k-1}{i-1} v_{n-rk+i}^{(r)} \\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} v_{n-r-r(k-1)+i}^{(r)} + \sum_{i=0}^{k-1} \binom{k-1}{i} v_{n-r-r(k-1)+i+1}^{(r)} \\ &= v_{n-r}^{(r)} + v_{n-r+1}^{(r)} = v_n^{(r)}, \text{ as required.}\end{aligned}$$

4. A DIVISIBILITY RESULT

If we refer to Table 1 again, we observe that 5 divides $v_{10}^{(4)}$, $v_{10}^{(5)}$, and $v_{10}^{(6)}$, 7 divides $v_{14}^{(4)}$, $v_{14}^{(5)}$, and $v_{14}^{(6)}$, etc. More generally, this can be expressed as

$$p \mid v_{pr}^{(r+n)} \text{ for } n > 1,\ r > 1, \text{ and prime } p > 2. \tag{4.1}$$

Proof:
$$\begin{aligned} v_{pr}^{(r+n)} &= \sum_{j=1}^{r+n} \alpha_j^{pr} \\ &= \sum_{j=1}^{r+n} ((\alpha_j + 1)/\alpha_j^2)^p \quad \text{from (2.1)} \\ &= \sum_{j=1}^{r+n} \sum_{k=0}^{p} \binom{p}{k} \alpha_j^{-p-k} \\ &= \sum_{j=1}^{r+n} (\alpha_j^{-p} + \alpha_j^{-2p}) + \text{multiples of } p.\end{aligned}$$

This follows from Hardy and Wright [2, p. 64] and the fact that

$$\binom{p}{k}\sum_{j=1}^{r+n}\alpha_j^{-p-k}$$

is an integer because $\Pi_{j=1}^{r+n}\alpha_j^{-p-k} = \pm 1$. It remains to show that

$$p\left|\sum_{j=1}^{r+n}(\alpha_j^{-p} + \alpha_j^{-2p}).\right.$$

The polynomial with zeros $1/\alpha_j$ is

$$f(x) = x^{r+n} + x^{r+n-1} - 1. \tag{4.2}$$

From the theory of equations, we have that

$$\begin{aligned} f'(x)/f(x) &= \sum_{j=1}^{r+n} 1/(x - x_j) \qquad \text{where } x_j = 1/\alpha_j \\ &= \sum_{j=1}^{r+n}\frac{1}{x}\left(1 - \frac{x_j}{x}\right)^{-1} \qquad \text{with } x_j < x. \end{aligned}$$

Thus

$$f'(x)/f(x) = \sum_{j=1}^{r+n}\sum_{m=0}^{\infty} x_j^m/x^{m+1} = \sum_{m=0}^{\infty} v_m^{(r+n)}/x^{m+1}. \tag{4.3}$$

Now, $f'(x) = (r + n)x^{r+n-1} + (r + n - 1)x^{r+n-2}$ and, by division,

$$f'(x)/f(x) = (r + n)x^{-1} - x^{-2} + x^{-3} - x^{-4} + x^{-5} - \cdots . \tag{4.4}$$

Since p is odd and $2p$ is even, we get from (4.3) and (4.4) that if $\Sigma_{j=1}^{r+n}\alpha_j^{-p} = -1$, then $\Sigma_{j=1}^{r+n}\alpha_j^{-2p} = +1$, and vice versa. Hence,

$$0 = \sum_{j=1}^{r+n}(\alpha_j^{-p} + \alpha_j^{-2p}), \text{ as required.}$$

5. CONCLUDING COMMENTS

The consideration of $v_n^{(r)}$ for $n < 0$ suggests the use of a result from Polyá and Szegö [9] to express the general term on the *negative* side of the sequence. Thus, for $n < 0$,

$$v_n^{(r)} = \sum_{k=0}^{[m/r]} \frac{m}{m - (r - 1)k}\binom{m - (r - 1)k}{k}(-1)^{m-rk}$$

in which $m = -n$, and $[\cdot]$ represents the greatest integer function.

The first few values are displayed in Table 2.

Table 2. Generalized Lucas Numbers for $n < 0$

$r \backslash n$	-9	-8	-7	-6	-5	-4	-3	-2	-1
2	-76	47	-29	18	-11	7	-4	3	-1
3	-7	5	-1	-2	4	-3	2	1	-1
4	-19	13	-8	7	-6	5	-1	1	-1

For example, when $r = 2$, we get the known result of Lucas [7]:

$$v_n^{(2)} = \sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k}(-1)^{n-1}. \tag{5.2}$$

The recurrence relation for (5.1) is (with $m = -n$)

$$(-1)^m v_m^{(r)} = (-1)^{m-1} v_{m-1}^{(r)} + (-1)^{m-r} v_{m-r}^{(r)}$$

so that these $v_m^{(r)} = (-1)^m A_m$ of Hock and McQuistan [3] who apply this sequence to a problem on the occupation statistics of lattice spaces in relation to a number of physical phenomena.

Other extensions can be found by developing an associated generalized Fibonacci sequence $\{u_n^{(r)}\}$, related to $\{v_n^{(r)}\}$ by, for instance

$$v_n^{(r)} = n \sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} u_{a_1}^{(r)} \ldots u_{a_k}^{(r)},$$

in which $\gamma(n)$ indicates summation over all the compositions $(a_1, a_2, \ldots, a_k)$ of n as in Shannon [11]. For example, when $r = 2$,

$$L_1 = 1 = f_1,$$
$$L_2 = 3 = -\frac{2}{2}f_1 f_1 + \frac{2}{1}f_2 = -1 + 4,$$
$$L_3 = 4 = -\frac{3}{2}f_1 f_2 - \frac{3}{2}f_2 f_1 + \frac{3}{1}f_3 + \frac{3}{3}f_1 f_1 f_1 = -3 - 3 + 9 + 1,$$

where $\{f_n\}$ is the sequence of Fibonacci numbers that satisfy (1.1) with initial conditions $f_1 = 1$, $f_2 = 2$. The use of the lower-case letters for notational convenience ($f_n \equiv F_{n+1}$) is not new (cf. [6]).

Thanks are due to Lambert Wilson [13] for the development of Table 1.

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