ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Assistant Editors
GLORIA C. PADILLA and CHARLES R. WALL

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Proposed problems should be accompanied by their solutions. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-544 *Proposed by Herta T. Freitag, Roanoke, VA*

Show that $F_{2n+1}^2 \equiv L_{2n+1}^2 \pmod{12}$ for all integers n.

B-545 Proposed by Herta T. Freitag, Roanoke, VA

Show that there exist integers a, b, and c such that

$$F_{4n} \equiv an \pmod 5 \quad \text{and} \quad F_{4n+2} \equiv bn + c \pmod 5$$

for all integers n.

B-546 *Proposed by Stuart Anderson, East Texas State University, Commerce, TX and John Corvin, Amoco Research, Tulsa, OK*

For positive integers a, let S_a be the finite sequence $a_1, a_2, \ldots, a_n$ defined by

$a_1 = a$,

$a_{i+1} = a_i/2$ if a_i is even, $a_{i+1} = 1 + a_i$ if a_i is odd,

the sequence terminates with the earliest term that equals 1.

For example, S_5 is the sequence 5, 6, 3, 4, 2, 1, of six terms. Let K_n be the number of positive integers a for which S_a consists of n terms. Does K_n equal something familiar?

B-547 *Proposed by Philip L. Mana, Albuquerque, NM*

For positive integers p and n with p prime, prove that

$$L_{np} \equiv L_n L_p \pmod{p}.$$

B-548 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Let $D(n)$ be defined inductively for nonnegative integers n by $D(0) = 0$ and $D(n) = 1 + D(n - [\sqrt{n}]^2)$, where $[x]$ is the greatest integer in x. Let n_k be the smallest n with $D(n) = k$. Then

$$n_0 = 0, \quad n_1 = 1, \quad n_2 = 2, \quad n_3 = 3, \quad \text{and} \quad n_4 = 7.$$

Describe a recursive algorithm for obtaining n_k for $k \geqslant 3$.

B-549 *Proposed by George N. Philippou, Nicosia, Cyprus*

Let $H_0, H_1, \ldots$ be defined by $H_0 = q - p$, $H_1 = p$, and $H_{n+2} = H_{n+1} + H_n$ for $n = 0, 1, \ldots$. Prove that, for $n \geqslant m \geqslant 0$,

$$H_{n+1}H_m - H_{m+1}H_n = (-1)^{m+1}[pH_{n-m+2} - qH_{n-m+1}].$$

SOLUTIONS

Coded Multiplication Modulo 10 or 12

B-520 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

(a) Suppose that one has a table for multiplication (mod 10) in which a, b, ..., j have been substituted for 0, 1, ..., 9 in some order. How many decodings of the substitution are possible?

(b) Answer the analogous question for a table of multiplication (mod 12).

Solution by the proposer.

(a) There are two ways to decode the substitution. The letters representing 0 and 1 are easy to find, since $x \cdot 0 = 0$ and $x \cdot 1 = x$ for all x; then 9 is easily found as the unique solution to $x^2 - 1$ with $x \neq 1$. The letter representing 5 is identifiable, and the letters are easily sorted as odd or even, because $5 \cdot x = 5$ if x is odd and $5 \cdot x = 0$ if x is even. Then 6 is identified from $6 \cdot x = x$ if x is even, and 4 is identified from $x^2 = 6$ with $x \neq 6$. Still unidentified are 2, 3, 7, and 8, but $2^2 = 8^2 = 4$ and $3^2 = 7^2 = 9$. so there are two choices for 3. Once 3 is chosen, 7 is forced, and so are 2 and 8, since $3 \cdot 4 = 2$ and $3 \cdot 6 = 8$.

(b) The substitution is unique. As in (a), 0 and 1 are easily identified. Then 6 is easily found, and the letters can be classified as odd or even, because $6 \cdot x = 6$ if x is odd and $6 \cdot x = 0$ if x is even. Now, 4 is the only nonzero even solution of $x^2 = x$. If x and y are both even, then $x \cdot y$ is 0, 4, or 8, and since 0 and 4 are already known, 8 is easily identified, leaving only 2 and 10 unknown among the even numbers. But $8 \cdot 2 = 4$ and $8 \cdot 10 = 8$, so 2 and 10 can be determined. Among the odd numbers, 9 is the only solution to $x^2 = x$ with $x \neq 1$, so 9 is easily identified. If x is odd, then $9 \cdot x$ is either 3 or 9, so 3 is determined. Then 7 is identified using the fact that $7 \cdot x = x$ if x is even. To identify 5 and 11, we use the fact that $3 \cdot 5 = 3$, while $3 \cdot 11 = 9$.

Also solved by Paul S. Bruckman and by L. Kuipers & P. A. J. Scheelbeek.

Unique Decoding

B-521 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

See the previous problem. Find all moduli $m > 1$ for which the multiplication (mod m) table can be decoded in only one way.

Solution by the proposer.

Suppose the multiplication (mod m) table can be decoded uniquely. Then it is easy to see that if $k|m$, the multiplication (mod k) table can also be decoded in only one way.

If $p \geqslant 5$ is prime, there are at least two distinct primitive roots (mod p), say g and h; replacing g^n by h^n for each n yields an equivalent substitution, so the multiplication (mod p) table cannot be decoded uniquely, and hence $p \nmid m$.

The multiplication (mod 9) table cannot be decoded uniquely, because 3 and 6 may be interchanged, and in the multiplication (mod 8) table, 2 and 6 may be switched.

Therefore, $m = 2^a 3^b$ with $a \leqslant 2$ and $b \leqslant 1$. Since the multiplication (mod 12) table can be decoded in only one way, $m = 2, 3, 4, 6$, or 12.

Also solved by Paul S. Bruckman and by L. Kiupers & P. A. J. Scheelbeek.

Alternating Even and Odd

B-522 *Proposed by Joan Tomescu, University of Bucharest, Romania*

Find the number $A(n)$ of sequences $(a_1, a_2, \ldots, a_k)$ of integers a_i satisfying $1 \leqslant a_i < a_{i+1} \leqslant n$ and $a_{i+1} - a_i \equiv 1 \pmod 2$ for $i = 1, 2, \ldots, k - 1$. [Here k is variable but, of course, $1 \leqslant k \leqslant n$. For example, the three allowable sequences for $n = 2$ are (1), (2), and (1, 2).]

Solution by J. Suck, Essen, Germany

$$A(n) = F_{n+3} - 2.$$

Proof by Double Induction

Let $B(n)$ be the number of sequences of the said type with $a_k = n$. I claim that $B(n) = F_{n+1}$. This is so for $n = 1, 2$. Suppose it is true for $\nu = 1, \ldots, n - 1 \geqslant 1$. The sequences with $a_k = n$, except (n), consist of those with $a_{k-1} = n - 1$ or $n - 3$ or $n - 5 \ldots$. Thus

$$B(m) = 1 + F_n + F_{n-2} + \cdots + \begin{cases} F_2 & \text{for } n \text{ even} \\ F_3 & \text{for } n \text{ odd} \end{cases}$$

$$= F_{n+1} \text{ in any case by Hoggatt's } I_5 \text{ or } I_6.$$

Now, $A(1) = 1 = F_4 - 2$, and, clearly,

$$A(n) = A(n - 1) + B(n) = F_{n+2} - 2 + F_{n+1} = F_{n+3} - 2 \quad \text{for } n > 1.$$

Also solved by Paul S. Bruckman, Laszlo Cseh, L. A. G. Dresel, Herta T. Freitag, L. Kuipers, J. Metzger, W. Moser, Sahib Singh, and the proposer.

Reversing Coefficients of a Polynomial

B-523 *Proposed by Laszlo Cseh and Imre Merenyi, Cluj, Romania*

Let $p, a_0, a_1, \ldots, a_n$ be integers with p a positive prime such that

$$\gcd(a_0, p) = 1 = \gcd(a_n, p).$$

Prove that in $\{0, 1, \ldots, p-1\}$ there are as many solutions of the congruence

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \equiv 0 \pmod{p}$$

as there are of the congruence

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \equiv 0 \pmod{p}.$$

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Since $\gcd(a_0, p) = \gcd(a_n, p) = 1$, it follows that both polynomials associated with the given congruences are of n^{th} degree and that zero is not a solution of any one of these congruences. If α is a solution of the first congruence, then α^{-1} is a solution of the second congruence where α^{-1} denotes the unique multiplicative inverse of α in Z_p.

Thus, we conclude that if $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the solutions of the first congruence in Z_p, then $\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_t^{-1}$ are precisely the solutions of the second congruence in Z_p.

Also solved by Paul S. Bruckman, L. A. G. Dresel, L. Kuipers, J. M. Metzger, Bob Prielipp, and the proposer.

Disguised Fibonacci Squares

B-524 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S_n = F_{2n-1}^2 + F_nF_{n-1}(F_{2n-1} + F_n^2) + 3F_nF_{n+1}(F_{2n-1} + F_nF_{n-1}).$$

Show that S_n is the square of a Fibonacci number.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let $a = F_n$, $b = F_{n-1}$. Note that $F_{2n-1} = a^2 + b^2$, $F_{n+1} = a + b$. Then

$$\begin{aligned} S_n &= (a^2 + b^2)^2 + ab(a^2 + b^2 + a^2) + 3a(a + b)(a^2 + b^2 + ab) \\ &= a^4 + 2a^2b^2 + b^4 + 2a^3b + ab^3 + 3a^4 + 6a^3b + 6a^2b^2 + 3ab^3 \\ &= 4a^4 + 8a^3b + 8a^2b^2 + 4ab^3 + b^4 \\ &= (2a^2 + 2ab + b^2)^2. \end{aligned}$$

Now $2a^2 + 2ab + b^2 = a^2 + (a + b)^2 = F_n^2 + F_{n+1}^2 = F_{2n+1}$. Hence, $S_n = F_{2n+1}^2$.

Also solved by L. A. G. Dresel, L. Kuipers, Imre Merenyi, J. M. Metzger, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Diophantine Equation

B-252 *Proposed by Walter Blumberg, Coral Springs, FL*

Let x, y, and z be positive integers such that $2^x - 1 = y^z$ and $x > 1$. Prove that $z = 1$.

Solution by Leonard A. G. Dresel, University of Reading, England

Since $x > 1$, we have $y^z = 2^x - 1 \equiv -1 \pmod 4$. Hence, $y \equiv -1 \pmod 4$ and z is odd, so that we have the identity

$$y^z + 1 = (y + 1)(y^{z-1} - y^{z-2} + \cdots - y + 1).$$

Hence, $y + 1$ divides $y^z + 1 = 2^x$, so that $y + 1 = 2^u$, $u \leqslant x$, and

$$\begin{aligned} 2^{x-u} &= y^{z-1} - y^{z-2} + \cdots - y + 1 \\ &\equiv 1 + 1 + \cdots + 1 + 1 \pmod 4 \\ &\equiv z \text{ (since there are } z \text{ terms)} \\ &\equiv 1 \text{ modulo } 2, \text{ since } z \text{ is odd.} \end{aligned}$$

Therefore, we must have $x - u = 0$, and $y^z = y$, and since $y^z > 1$ it follows that $z = 1$.

Note by Paul S. Bruckman

This is apparently a well-known result, indicated by S. Ligh and L. Neal in "A Note on Mersenne Numbers," *Math. Magazine* 47, no. 4 (1974):231-33. The result indicated in that reference is that a Mersenne number cannot be a power (greater than one) of an integer.

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, Laszlo Cseh, L. Kuipers, Imre Merenyi, J. M. Metzger, Bob Prielipp, E. Schmutz & H. Klauser, Sahib Singh, J. Suck, and the proposer.

◆◇◆◇◆