A CHARACTERIZATION OF THE SECOND-ORDER STRONG DIVISIBILITY SEQUENCES

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The Fibonacci numbers satisfy the well-known equation for greatest common divisors (cf. [2], [4]):

$$(F_i, F_j) = F_{(i,j)} \quad \text{for all } i, j \geq 1. \tag{1}$$

Equation (1) is also satisfied by some other second-order recurring sequences of integers, e.g., Pell numbers or Fibonacci polynomials evaluated at a fixed integer (cf. [1]). In [3], Clark Kimberling put a question: Which recurrent sequences satisfy the equation (1)? In our paper, we answer this question for a certain class of recurring sequences, namely that of the second-order linear recurrent sequences of integers.

We shall study the sequences $\mathbf{u} = \{u_n : n = 1, 2, \ldots\}$ of integers defined by

$$u_1 = 1, \quad u_2 = b, \quad u_{n+2} = c \cdot u_{n+1} + d \cdot u_n, \quad \text{for } n \geq 1,$$

where b, c, d are arbitrary integers. The system of all such sequences will be denoted by U. The system of all the sequences from U, having the property

$$(u_i, u_j) = |u_{(i,j)}| \quad \text{for all } i, j \geq 1, \tag{2}$$

will be denoted by D.

The main result of our paper is a complete characterization of all sequences from D. By describing D we solve, in fact, a more general problem of complete characterization of all the second-order, strong-divisibility sequences, i.e., all sequences $\{u_n\}$ of integers defined by

$$u_1 = a, \quad u_2 = b, \quad u_{n+2} = c \cdot u_{n+1} + d \cdot u_n, \quad \text{for } n \geq 1,$$

(where a, b, c, d are arbitrary integers) and satisfying equation (2). It is easy to prove that the second-order, strong-divisibility sequences are precisely all integral multiples of sequences from D.

1. CERTAIN SYSTEMS OF SEQUENCES FROM U

Systems U_1, F, F_1, G, G_1, H will be systems of all sequences $\mathbf{u} = \{u_n\}$ from U defined by $u_1 = 1$, $u_2 = b$, and by the recurrence relations (for $n \geq 1$):

$$U_1 : u_{n+2} = b \cdot f \cdot u_{n+1} + d \cdot u_n, \quad \text{where } b, d, f \neq 0, F \neq 1, \; (d, b) = (d, f) = 1;$$

$$F : u_{n+2} = b \cdot u_{n+1} + d \cdot u_n;$$

$$F_1 : u_{n+2} = b \cdot u_{n+1} + d \cdot u_n, \quad \text{where } (d, b) = 1;$$

$$G : u_{n+2} = d \cdot u_n;$$

$$G_1 : u_{n+2} = d \cdot u_n, \quad \text{where } d = 1 \text{ or } d = -1;$$

$$H : u_{n+2} = c \cdot u_{n+1}.$$

It is obvious that $F_1 \subseteq F$ and $G_1 \subseteq G$. Further, we define sequences **a**, **b**, **c**, **d**, **e**, **f** $= \{u_n\} \in U$ by:

$$\mathbf{a} : u_n = 1 \text{ for all } n \geqslant 1 \qquad \mathbf{b} : u_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

$$\mathbf{c} : u_n = \begin{cases} 1 & \text{if } n = 1 \\ -1 & \text{if } n > 1 \end{cases} \qquad \mathbf{d} : u_n = \begin{cases} 1 & \text{if } n = 1 \text{ or } n \text{ is even} \\ -1 & \text{if } n \neq 1 \text{ and } n \text{ is odd} \end{cases}$$

$$\mathbf{e} : u_n = \begin{cases} 1 & \text{if } 3 \nmid n \\ -2 & \text{if } 3 \mid n \end{cases} \qquad \mathbf{f} : u_n = \begin{cases} 1 & \text{if } n \equiv 1, 5 \pmod 6 \\ -1 & \text{if } n \equiv 2, 4 \pmod 6 \\ -2 & \text{if } n \equiv 3 \pmod 6 \\ 2 & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

Let us denote $A = \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. Directly from the definitions we obtain:

1.1 Proposition

1. **a**, **b**, **c**, **d**, **e**, **f** $\in D$, i.e., $A \subseteq D$
2. **a**, **b**, **c**, **d** $\in H$
3. **a**, **b**, **e**, **f** $\in U_1$
4. **a**, **b** $\in F_1 \cap G_1$

1.2 Proposition

Let **u** $= \{u_n\} \in G$. Then **u** $\in D$ if and only if **u** $\in G_1$.

Proof: Let **u** $\in D$; then $(u_3, u_4) = 1$ and consequently **u** $\in G_1$. Let **u** $\in G_1$; then for $k \geqslant 0$ we get $u_{4k+1} = 1$, $u_{4k+2} = b$, $u_{4k+3} = \pm 1$, $u_{4k+4} = \pm b$. Thus, for $i, j \geqslant 1$,

$$(u_i, u_j) = \begin{cases} 1 & \text{if } i \text{ is odd or } j \text{ is odd} \\ |b| & \text{if } i \text{ is even and } j \text{ is even} \end{cases}$$

and therefore, **u** $\in D$.

1.3 Proposition

Let **u** $= \{u_n\} \in H$. Then **u** $\in D$ if and only if **u** $\in \{$**a**, **b**, **c**, **d**$\}$.

Proof: Let **u** $\in D$; then $(u_2, u_3) = (u_3, u_4) = 1$ and we get $|b| = 1$, $|c| = 1$, and consequently **u** $\in \{$**a**, **b**, **c**, **d**$\}$. The rest of the proposition follows from 1.1.

1.4 Proposition

Let **u** $= \{u_n\} \in U$, such that $c, d \neq 0$. Then, the following statements are equivalent:

(i) $(u_i, u_j) = |u_{(i,j)}|$ for $1 \leqslant i, j \leqslant 4$,

(ii) **u** $\in U_1 \cup F_1$.

Proof: Let (i) be true. From $(u_2, u_3) = 1$ we get $(b, d) = 1$. From $u_2 | u_4$ we get $b|c$, $b \neq 0$. Therefore, there is an integer $f \neq 0$, such that $c = bf$. Since $(u_3, u_4) = 1$, we have $(d, f) = 1$ and thus $\mathbf{u} \in U_1 \cup F_1$.

Let $\mathbf{u} \in U_1 \cup F_1$. Then $u_3 = d + b^2f$, $u_4 = b(d + df + b^2f^2)$, where $b, f \neq 0$, $(d, b) = (d, f) = 1$. Let p be a prime, $p|u_3$ and $p|u_4$. Obviously $p \nmid b$, and so $d + b^2f \equiv 0 \pmod p$, $d + df + b^2f^2 \equiv 0 \pmod p$. Hence $b^2f \equiv 0 \pmod p$ and consequently $p|f$, $p|d$, a contradiction. Thus $(u_3, u_4) = 1 = |u_1|$. The remaining cases of (i) obviously hold.

2. THE SYSTEM OF SEQUENCES F

The following two results are easily proved by mathematical induction, in the same way as for the Fibonacci numbers (cf. [4]).

2.1 Proposition

Let $\mathbf{u} = \{u_n\} \in F$. Then for any $k \geqslant 2$, $m \geqslant 1$ it holds

$$u_{k+m} = u_k u_{m+1} + d \cdot u_{k-1} u_m.$$

2.2 Proposition

Let $\mathbf{u} = \{u_n\} \in F$ and $k, m \geqslant 1$ be integers. If $k|m$, then $u_k | u_m$.

2.3 Proposition

Let $\mathbf{u} = \{u_n\} \in F$. Then the following statements are equivalent.

(i) $(u_2, u_3) = 1$

(ii) $(u_n, u_{n+1}) = 1$ for all $n \geqslant 1$

(iii) $\mathbf{u} \in D$

(iv) $\mathbf{u} \in F_1$

Proof: Clearly (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv). Let (iv) be true. Let r be the smallest positive integer such that $(u_r, u_{r+1}) \neq 1$. Then $r \geqslant 2$ and there exists a prime p such that $p|u_r$, $p|u_{r+1}$. But $u_{r+1} = bu_r + du_{r-1}$, and hence $p|d$. Now, it is easy to prove, by induction, that $u_n \equiv b^{n-1} \pmod p$, for all $n \geqslant 1$. Hence, $0 \equiv u_r \equiv b^{r-1} \pmod p$ so that $p|b$, a contradiction, and (iv) $\Rightarrow$ (ii) is proved.

Now, let (ii) be true. We can assume that $i > j > 1$. Let $g = (u_i, u_j)$. Then from 2.2 we get $u_{(i,j)} | g$. It is well known that there exist integers r, s with, say, $r > 0$ and $s < 0$, such that $(i, j) = ri + sj$. Thus, by 2.1, we get

$$u_{ri} = u_{(-s)j + (i,j)} = u_{(-s)j} u_{(i,j)+1} + du_{(-s)j-1} u_{(i,j)}.$$

But by 2.2, $g | u_{(-s)j}$, $g | u_{ri}$, and by (ii), $(g, u_{(-s)j-1}) = 1$, so that $g | du_{(i,j)}$. If p is a prime, $p|g$, $p|d$, then $p | u_i = bu_{i-1} + du_{i-2}$, and so $p|b$. Thus, $(u_2, u_3) > 1$, a contradiction. Hence, $(g, d) = 1$ so that $g | u_{(i,j)}$ and (iii) is true.

3. THE SYSTEM OF SEQUENCES U_1

If $\mathbf{u} = \{u_n\} \in U_1$, then directly from the definition we obtain

$$u_3 = d + b^2 f, \quad u_4 = b(d + df + b^2 f^2)$$

and

$$u_5 = d^2 + 2b^2df + b^2df^2 + b^4f^3, \tag{3}$$

where b, d, $f \neq 0$, $f \neq 1$, and $(d, b) = (d, f) = 1$.

3.1 Proposition

Let $\mathbf{u} = \{u_n\} \in U_1$. Then the following statements are equivalent.

(i) $u_3 | u_6$

(ii) $u_3 \neq 0$ and $f \equiv 1 \pmod{|u_3|}$

Proof:

I: Let $u_3 | u_6$ and let $0 = u_3 = d + b^2f$. Then $u_6 = bd(d + b^2f^2) = 0$, and consequently, $f = 1$, a contradiction. Thus, from (i), it follows that $u_3 \neq 0$.

II: Let $u_3 \neq 0$. Since $u_6 \equiv bd(d + b^2f^2) \pmod{|u_3|}$ and $(bd, u_3) = 1$, we have $u_3 | u_6$ iff $d + b^2f^2 \equiv 0 \pmod{|u_3|}$. But $d + b^2f^2 \equiv b^2f(-1 + f) \pmod{|u_3|}$, and $(f, u_3) = (b, u_3) = 1$, so that $d + b^2f^2 \equiv 0 \pmod{|u_3|}$ iff $f \equiv 1 \pmod{|u_3|}$.

3.2 Proposition

Let $\mathbf{u} = \{u_n\} \in U_1$. Then the following statements are equivalent.

(i) $u_4 | u_8$

(ii) $d + df + b^2f^2 \neq 0$ and $f \equiv 1 \pmod{|d + df + b^2f^2|}$

Proof:

I: Let $u_4 | u_8$ and $d + df + b^2f^2 = 0$. Then

$$u_4 = 0 \quad \text{and} \quad u_8 = bdf(2d + b^2f^2)u_3 = 0.$$

But both $2d + b^2f^2 = 0$ and $u_3 = 0$ lead immediately to a contradiction; thus, from (i) it follows that $d + df + b^2f^2 \neq 0$.

II: Let $d + df + b^2f^2 \neq 0$. Clearly, $u_8 \equiv bdf(2d + b^2f^2)u_3 \pmod{|u_4|}$ and $(df, d + df + b^2f^2) = 1$, and, from 1.4, we get $(u_3, u_4) = 1$. Hence, $u_4 | u_8$ iff $2d + b^2f^2 \equiv 0 \pmod{|d + df + b^2f^2|}$. Trivially,

$$b^2f^2 \equiv -d - df \pmod{|d + df + b^2f^2|}$$

and thus,

$$2d + b^2f^2 \equiv 0 \pmod{|d + df + b^2f^2|} \text{ iff } f \equiv 1 \pmod{|d + df + b^2f^2|}.$$

3.3 Lemma

Let $b, d, f \neq 0$, $f \neq 1$ be integers such that $(d, b) = (d, f) = 1$, $d + b^2f \neq 0$, and $d + df + b^2f^2 \neq 0$.

Then $f \equiv 1 \pmod{|d + b^2f|}$ and $f \equiv 1 \pmod{|d + df + b^2f^2|}$ if and only if one of the following cases occurs:

$b = \pm 1$, $f = -1$, $d = -1$	$b = \pm 1$, $f = -2$, $d = 1, 5$
$b = \pm 1$, $f = -3$, $d = 5$	$b = \pm 1$, $f = -5$, $d = 7$
$b = \pm 1$, $f + d = 1$	$f = \pm b^2$, $d = \mp 1 + b^2 \mp b^4$

Proof: Sufficiency is easy to verify in all of the cases, so we prove necessity. Let us denote $x = d + b^2f$, $y = d + df + b^2f^2$. Clearly, $(x, y) = 1$, $x \equiv y \pmod{|f|}$, and

$$y = x + fx - b^2f \tag{4}$$

$$xy \mid (f - 1). \tag{5}$$

α) Suppose $f > 1$.

Then $x \equiv y \pmod{f}$, and from (5) we get $|x|, |y| < f$.

α_1) If $x, y > 0$ or $x, y < 0$, then $x = y$, and hence $b^2 = d + b^2f^2$. So $b \mid d$ and we get $b = \pm 1$, $f + d = 1$.

α_2) $x < 0$, $y > 0$ is impossible because of (4).

α_3) If $x > 0$, $y < 0$, then $y = x - f$, where $0 < x < f$.

From (4) we get $x = b^2 - 1$ and from (5) we get $x(f - x) \mid f - 1$. If $x \leqslant (f - 1)/2$, then $f - x > (f - 1)/2$, and hence $f - x = f - 1$. But then $x = 1 = b^2 - 1$, a contradiction. If $x > (f - 1)/2$, then $x = f - 1$. Thus, we get $f = b^2$, $d = -1 + b^2 - b^4$.

β) Suppose $f < 0$.

Denote $t = -f$. Then $x \equiv y \pmod{t}$, and from (5) we get $|x|, |y| \leqslant t + 1$.

β_1) If $|x| = t + 1$ or $|y| = t + 1$, then there are four possibilities:

β_{11}) $x = f - 1$, $y = \pm 1 = f^2 - b^2f - 1$.

From $1 = f^2 - b^2f - 1$, we get $b = \pm 1$, $f = -1$, $d = -1$, and $-1 = f^2 - b^2f - 1$ is impossible, since then we get $f = b^2 > 0$, a contradiction.

β_{12}) $x = -(f - 1)$, $y = \pm 1 = -f^2 - b^2f + 1$.

From $1 = -f^2 - b^2f + 1$, we get $f = -b^2$, $d = 1 + b^2 + b^4$, and from $-1 = -f^2 - b^2f + 1$, we get $b = \pm 1$, $f = -2$, $d = 5$.

β_{13}) $x = \pm 1$, $y = f - 1 = \pm 1 \pm f - b^2f$ both lead to a contradiction.

β_{14}) $x = \pm 1$, $y = -(f - 1) = \pm 1 \pm f - b^2f$.

From $-f + 1 = 1 + f - b^2f$, we get $b^2 = 2$, a contradiction, and from $-f + 1 = -1 - f - b^2f$, we get $b = \pm 1$, $f = -2$, $d = 1$.

β_2) If $|x| = t$ or $|y| = t$ and $|x|, |y| \neq t + 1$, then $t \mid t + 1$, and hence $f = -1$. We get $b = \pm 1$, $f = -1$, $d = 2$, which is a special case of $b = \pm 1$, $f + d = 1$.

β_3) If $|x|, |y| < t$, then we have the following possibilities:

β_{31}) $x, y > 0$ or $x, y < 0$. Then $x = y$, and in the same way as in α_1), we get $b = \pm 1$, $f + d = 1$.

β_{32}) $x < 0$, $y > 0$ is impossible because, then, $x = y + f$, and we get $y = b^2 - f - 1$, so that $x = b^2 - 1 \geqslant 0$, a contradiction.

β_{33}) $x > 0$, $y < 0$. Then $y = x - t = x + f$, and hence $x = b^2 + 1$. From (5), we get $x(t - x) \mid t + 1$, where $0 < x < t$ and $0 < t - x < t$.

If $x < (t + 1)/2$, then $t - x > (t - 1)/2$. From $t - x = t/2$, we get a contradiction, and hence $t - x = (t + 1)/2$, $x = (t - 1)/2$. Now, from $(t - 1)/2 \cdot (t + 1)/2 \mid t + 1$, we get $(t - 1)/2 \mid 2$, and consequently $b = \pm 1$, $f = -5$, $d = 7$. If $x \geqslant (t + 1)/2$, then, similarly as above, we get $b = \pm 1$, $f = -3$, $d = 5$.

3.4 Proposition

Let $\mathbf{u} = \{u_n\} \in U_1$. Then the following statements are equivalent.

(i) $u_5 | u_{10}$

(ii) $u_5 \neq 0$ and $d^2 + 3b^2df^2 + b^4f^4 \equiv 0 \pmod{|u_5|}$

Proof:

I: Let $u_5 | u_{10}$ and $0 = u_5 = d^2 + 2b^2df + b^2df^2 + b^4f^3$. Then $u_{10} = d(d^2 + 3b^2df^2 + b^4f^4)u_4 = 0$. If $u_4 = 0$, then $0 = d + df + b^2f^2 = d(1 + f) + b^2f^2$ and from (3) we get $u_5(1 + f)^2 = -b^4f^3 \neq 0$, a contradiction. Thus, we have $d^2 + 3b^2df^2 + b^4f^4 = 0$. But then $d^2 = -3b^2df^2 - b^4f^4$ and from $0 = u_5$ we get $b^2f^2 = -2d$, which is a contradiction, since $(d, b) = (d, f) = 1$.

II: Let $u_5 \neq 0$. Then

$u_{10} \equiv d(d^2 + 3b^2df^2 + b^4f^4)u_4 \pmod{|u_5|}$.

It is easy to prove that $(u_4, u_5) = 1$ and $(d, u_5) = 1$. Thus, $u_5 | u_{10}$ if and only if $d^2 + 3b^2df^2 + b^4f^4 \equiv 0 \pmod{|u_5|}$.

3.5 Proposition

Let $\mathbf{u} = \{u_n\} \in U_1$. Then the following statements are equivalent.

(i) $u_3 | u_6$, $u_4 | u_8$, $u_5 | u_{10}$

(ii) $\mathbf{u} \in D$

(iii) $\mathbf{u} \in \{\mathbf{a}, \mathbf{b}, \mathbf{e}, \mathbf{f}\}$

Proof: Clearly (iii) ⇒ (ii) and (ii) ⇒ (i). Let (i) be true. According to 3.1 and 3.2, just the cased described in 3.3 can occur for the integers b, d, f.

α) If $b = 1$, $f + d = 1$, then $\mathbf{u} = \mathbf{a}$;
If $b = -1$, $f + d = 1$, then $\mathbf{u} = \mathbf{b}$;
If $b = 1$, $f = -1$, $d = -1$, then $\mathbf{u} = \mathbf{e}$;
If $b = -1$, $f = -1$, $d = -1$, then $\mathbf{u} = \mathbf{f}$.

β) If $f = b^2$, $d = -1 + b^2 - b^4$, then

$u_5 = -b^6 + b^4 - 2b^2 + 1$

and

$$d^2 + 3b^2df^2 + b^4f^4 = b^{12} - 3b^{10} + 4b^8 - 5b^6 + 3b^4 - 2b^2 + 1$$
$$= (-b^6 + b^4 - 2b^2 + 1)(-b^6 + 2b^4 + 1) + b^6.$$

Obviously, $(-b^6 + b^4 - 2b^2 + 1, b^6) = 1$ for every integer b. So, from 3.4, we get $-b^6 + b^4 - 2b^2 + 1 = \pm 1$, and thus $1 = b^2 = f$, a contradiction.

γ) If $f = -b^2$, $d = 1 + b^2 + b^4$, then

$u_5 = b^6 + b^4 + 2b^2 + 1$

and

$$d^2 + 3b^2df^2 + b^4f^4 = b^{12} + 3b^{10} + 4b^8 + 5b^6 + 3b^4 + 2b^2 + 1$$
$$= (b^6 + b^4 + 2b^2 + 1)(b^6 + 2b^4) + b^4 + 2b^2 + 1.$$

But $b^6 + b^4 + 2b^2 + 1 > b^4 + 2b^2 + 1 > 0$ for every nonzero integer b, which contradicts 3.4.

γ) It is easy to prove by direct calculation that the remaining cases of Lemma 3.3 also contradict 3.4.

4. MAIN THEOREM

4.1 Theorem

It holds that $D = A \cup F_1 \cup G_1$.

Proof:

I: Let $\mathbf{u} \in D$. If $c, d \neq 0$ then, by 1.4, 3.5, and 1.1.4, $\mathbf{u} \in F_1$ or $\mathbf{u} \in A$; if $c = 0$, then $\mathbf{u} \in G$ and, by 1.2, $\mathbf{u} \in G_1$; if $d = 0$, then $\mathbf{u} \in H$ and, by 1.3 and 1.1.4, $\mathbf{u} \in F_1$ or $\mathbf{u} \in A$. Hence, $\mathbf{u} \in A \cup F_1 \cup G_1$.

II: Let $\mathbf{u} \in A \cup F_1 \cup G_1$. Then, by 1.1.1, 2.3, and 1.2, we get $\mathbf{u} \in D$.

4.2 Corollary

All the second-order, strong-divisibility sequences are precisely all integral multiples of sequences from D, i.e., of the following sequences:

$\mathbf{c} = \{1, -1, -1, -1, \ldots\}$

$\mathbf{d} = \{1, 1, -1, 1, -1, \ldots\}$

$\mathbf{e} = \{1, 1, -2, 1, 1, -2, \ldots\}$

$\mathbf{f} = \{1, -1, -2, -1, 1, 2, 1, -1, -2, -1, 1, 2, \ldots\}$

$u_1 = 1, \quad u_2 = b, \qquad u_{n+2} = b \cdot u_{n+1} + d \cdot u_n$ where $(d, b) = 1$

$u_1 = 1, \quad u_2 = b, \qquad u_{n+2} = d \cdot u_n$ where $d = \pm 1$.

4.3 Remark

It is easy to prove that the systems A, F_1, G_1 satisfy

$A \cap F_1 = \phi, \quad A \cap G_1 = \phi, \quad F_1 \cap G_1 = \{\mathbf{a}, \mathbf{b}, \mathbf{g}, \mathbf{h}\}$,

where $g = \{1, 0, 1, 0, \ldots\}$, and $h = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$.

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