# FIBONACCI-TYPE POLYNOMIALS OF ORDER $K$ WITH PROBABILITY APPLICATIONS

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## 1. INTRODUCTION AND SUMMARY

In this paper, $k$ is a fixed integer greater than or equal to 2, unless otherwise stated, $n_i$ $(1 \leqslant i \leqslant k)$ and $n$ are nonnegative integers as specified, $p$ and $x$ are real numbers in the intervals $(0, 1)$ and $(0, \infty)$, respectively, and $[x]$ denotes the greatest integer in $x$. Set $q = 1 - p$, let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order $k$ [4], and denote by $N_k$ the number of Bernoulli trials until the occurrence of the $k^{\text{th}}$ consecutive success. We recall the following results of Philippou and Muwafi [4] and Philippou [3]:

$$P(N_k = n + k) = p^{n+k} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant 0; \tag{1.1}$$

$$f_{n+1}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geqslant 0; \tag{1.2}$$

$$f_{n+1}^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n - ki}{i} 2^{-(k+1)i} - 2^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n - 1 - ki}{i} 2^{-(k+1)i}, \quad n \geqslant 1. \tag{1.3}$$

For $p = 1/2$, (1.1) reduces to

$$P(N_k = n + k) = f_{n+1}^{(k)} / 2^{n+k}, \quad n \geqslant 0, \tag{1.4}$$

which relates probability to the Fibonacci sequence of order $k$. Formula (1.4) appears to have been found for the first time by Shane [8], who also gave formulas for $P(N_k = n)$ $(n \geqslant k)$ and $P(N_k \leqslant x)$, in terms of his polynacci polynomials of order $k$ in $p$. Turner [9] also derived (1.4) and found another general formula for $P(N_k = n + k)$ $(n \geqslant 0)$, in terms of the entries of the Pascal-$T$ triangle. None of the above-mentioned references, however, addresses the question of whether $\{P(N_k = n + k)\}_{n=0}^{\infty}$ is a proper probability distribution (see Feller [1, p. 309]), and none includes any closed formula for $P(N_k \leqslant x)$.

Motivated by the above results and open questions, we presently introduce a simple generalization of $\{f_n^{(k)}\}_{n=0}^{\infty}$, denoted by $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ and called a sequence of Fibonacci-type polynomials of order $k$, and derive appropriate analogs of (1.2)-(1.4) for $F_n^{(k)}(x)$ $(n \geqslant 1)$ [see Theorem 2.1 and Theorem 3.1(a)]. In addition, we show that $\sum_{n=0}^{\infty} P(N_k = n + k) = 1$, and derive a simple and closed formula for the distribution function of $N_k$ [see Theorem 3.1(b)-(c)].

## 2. FIBONACCI-TYPE POLYNOMIALS OF ORDER $K$ AND MULTINOMIAL EXPANSIONS

In this section, we introduce a sequence of Fibonacci-type polynomials of order $k$, denoted by $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$, and derive two expansions of $F_n^{(k)}(x)$ $(n \geq 1)$ in terms of the multinomial and binomial coefficients, respectively. The proofs are given along the lines of [3], [5], and [7].

### Definition 2.1

The sequence of polynomials $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequence of Fibonacci-type polynomials of order $k$, if

$$F_0^{(k)}(x) = 0,$$
$$F_1^{(k)}(x) = 1,$$

and

$$F_n^{(k)}(x) = \begin{cases} x[F_{n-1}^{(k)}(x) + \cdots + F_0^{(k)}(x)], & \text{if } 2 \leq n \leq k, \\ x[F_{n-1}^{(k)}(x) + \cdots + F_{n-k}^{(k)}(x)], & \text{if } n \geq k+1. \end{cases}$$

It follows from the definition of $\{f_n^{(k)}\}_{n=0}^{\infty}$ and Definition 2.1 that

$$F_n^{(k)}(1) = f_n^{(k)} \quad (n \geq 0).$$

The $n^{\text{th}}$ term of the sequence $\{F_n^{(k)}(x)\}$ $(n \geq 1)$ may be expanded as follows:

### Theorem 2.1

Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order $k$. Then

$$\text{(a)} \quad F_{n+1}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} x^{n_1 + \cdots + n_k}, \quad n \geq 0;$$

$$\text{(b)} \quad F_{n+1}^{(k)}(x) = (1+x)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n - ki}{i} x^i (1+x)^{-(k+1)i}$$
$$- (1+x)^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} x^i (1+x)^{-(k+1)i}, \quad n \geq 1.$$

We shall first establish the following lemma:

### Lemma 2.1

Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order $k$, and denote its generating function by $G_k(s; x)$. Then, for $|s| < 1/(1+x)$,

$$G_k(s; x) = \frac{s - s^2}{1 - (1+x)s + xs^{k+1}} = \frac{s}{1 - xs - xs^2 - \cdots - xs^k}.$$

Proof: We see from Definition 2.1 that

$$F_n^{(k)}(x) = \begin{cases} x(1+x)^{n-2}, & 2 \leqslant n \leqslant k+1, \\ (1+x)F_{n-1}^{(k)}(x) - xF_{n-1-k}^{(k)}(x), & n \geqslant k+2. \end{cases} \tag{2.1}$$

By induction on $n$, the above relation implies $F_n^{(k)}(x) \leqslant x(1+x)^{n-2}$ $(n \geqslant 2)$, which shows the convergence of $G_k(s;x)$ for $|s| < 1/(1+x)$. Next, by means of (2.1), we have

$$G_k(s;\, x) = \sum_{n=0}^{\infty} s^n F_n^{(k)}(x) = s + \sum_{n=2}^{k+1} s^n x(1+x)^{n-2} + \sum_{n=k+2}^{\infty} s^n F_n^{(k)}(x)$$

and

$$\sum_{n=k+2}^{\infty} s^n F_n^{(k)}(x) = (1+x)\sum_{n=k+2}^{\infty} s^n F_{n-1}^{(k)}(x) - x\sum_{n=k+2}^{\infty} s^n F_{n-1-k}^{(k)}(x)$$
$$= [(1+x)s - xs^{k+1}]G_k(s;\, x) - s^2 - \sum_{n=2}^{k+1} s^n x(1+x)^{n-2},$$

from which the lemma follows.

<u>Proof of Theorem 2.1</u>

First we shall show part (a). Let $|s| < 1/(1+x)$. Then, using Lemma 2.1 and the multinomial theorem, and replacing $n$ by $n - \sum_{i=1}^{k}(i-1)n_i$, we get

$$\sum_{n=0}^{\infty} s^n F_{n+1}^{(k)}(x) = \sum_{n=0}^{\infty}(xs + xs^2 + \cdots + xs^k)^n$$
$$= \sum_{n=0}^{\infty} \sum_{\substack{n_1,\, \ldots,\, n_k \ni \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1,\, \ldots,\, n_k} x^{n_1 + \cdots + n_k} s^{n_1 + 2n_2 + \cdots + kn_k}$$
$$= \sum_{n=0}^{\infty} s^n \sum_{\substack{n_1,\, \ldots,\, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1,\, \ldots,\, n_k} x^{n_1 + \cdots + n_k},\ n \geqslant 0,$$

which shows (a).

We now proceed to part (b). Set

$$A_k(x) = \{s \in R;\ |s| < 1/(1+x) \text{ and } |(1+x)s - xs^{k+1}| < 1\},$$

and let $s \in A_k(x)$. Then, using Lemma 2.1 and the binomial theorem, replacing by $n - ki$, and setting

$$b_n^{(k)}(x) = (1+x)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^i (1+x)^{-(k+1)i},\ n \geqslant 0,$$

we get

$$\sum_{n=0}^{\infty} s^n F_{n+1}^{(k)}(x) = (1-s)\sum_{n=0}^{\infty}[(1+x)s - xs^{k+1}]^n$$
$$= (1-s)\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(1+x)^{n-i}x^i s^{n+ki}$$
$$= (1-s)\sum_{n=0}^{\infty} s^n \sum_{i=0}^{[n/(k+1)]}(-1)^i\binom{n-ki}{i}(1+x)^{n-(k+1)i}x^i$$
$$= (1-s)\sum_{n=0}^{\infty} s^n b_n^{(k)}(x) = 1 + \sum_{n=1}^{\infty} s^n[b_n^{(k)}(x) - b_{n-1}^{(k)}(x)].$$

The last two relations establish part (b).

## 3. FIBONACCI-TYPE POLYNOMIALS OF ORDER $K$ AND PROBABILITY APPLICATIONS

In this section we shall establish the following theorem which relates the Fibonacci-type polynomials of order $k$ to probability, shows that

$$\{P(N_k = n + k)\}_{n=0}^{\infty}$$

is a proper probability distribution, and gives the distribution function of $N_k$.

Theorem 3.1

Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order $k$, denote by $N_k$ the number of Bernoulli trials until the occurrence of the $k^{\text{th}}$ consecutive success, and set $q = 1 - p$. Then

(a) $P(N_k = n + k) = p^{n+k}F_{n+1}^{(k)}(q/p), \; n \geqslant 0;$

(b) $\displaystyle\sum_{n=0}^{\infty} P(N_k = n + k) = 1;$

(c) $$P(N_k \leqslant x) = \begin{cases} 1 - \dfrac{p^{[x]+1}}{q} \displaystyle\sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = [x]+1}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}\left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, & x \geqslant k, \\ 0, \text{ otherwise.} \end{cases}$$

We shall first establish the following lemma.

Lemma 3.1

Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order $k$. Then, for any fixed $x \in (0, \infty)$,

(a) $$\lim_{n\to\infty} \frac{F_n^{(k)}(x)}{(1 + x)^n} = 0;$$

(b) $$\sum_{n=0}^{m} \frac{F_{n+1}^{(k)}(x)}{(1 + x)^{n+k}} = 1 - \frac{F_{m+k+2}^{(k)}(x)}{(1 + x)^{m+k}}, \quad m \geqslant 0.$$

Proof: First, we show (a). For any fixed $x \in (0, \infty)$ and $n \geqslant k + 1$, relation (2.1) gives

$$\frac{F_n^{(k)}(x)}{(1 + x)^n} - \frac{F_{n+1}^{(k)}(x)}{(1 + x)^{n+1}} = \frac{(1 + x)F_n^{(k)}(x) - F_{n+1}^{(k)}(x)}{x(1 + x)^{n+1}} = \frac{xF_{n-k}^{(k)}(x)}{(1 + x)^{n+1}} > 0,$$

which implies that $F_n^{(k)}(x)/(1 + x)^n$ converges. Therefore,

$$\lim_{n\to\infty} \frac{xF_{n-k}^{(k)}(x)}{(1+x)^{n+1}} = 0,$$

from which (a) follows.

We now proceed to show (b). For $m = 0$, both the left- and right-hand sides equal $(1+x)^{-k}$, since $F_{k+2}^{(k)}(x) = x(1+x)^k - x$ by (2.1). We assume that the lemma holds for some integer $m \geqslant 1$ and we shall show that it is true for $m+1$. In fact,

$$\sum_{n=0}^{m+1} \frac{F_{n+1}^{(k)}(x)}{(1+x)^{n+k}} = \frac{F_{m+2}^{(k)}(x)}{(1+x)^{m+k+1}} + \sum_{n=0}^{m} \frac{F_{n+1}^{(k)}(x)}{(1+x)^{n+k}}$$

$$= \frac{F_{m+2}^{(k)}(x)}{(1+x)^{m+1+k}} + 1 - \frac{F_{m+k+2}^{(k)}(x)}{x(1+x)^{m+k}}, \text{ by induction hypothesis,}$$

$$= 1 - \frac{(1+x)F_{m+k+2}^{(k)}(x) - xF_{m+2}^{(k)}(x)}{x(1+x)^{m+k+1}}$$

$$= 1 - \frac{F_{m+k+3}^{(k)}(x)}{x(1+x)^{m+k+1}}, \text{ by (2.1).}$$

Proof of Theorem 3.1

Part (a) follows directly from relation (1.1), by means of Theorem 2.1 applied with $x = q/p$. Next, we observe that

$$\sum_{n=0}^{m} P(N_k = n+k) = \sum_{n=0}^{m} p^{n+k}F_{n+1}(q/p), \text{ by Theorem 3.1(a),}$$

$$= \sum_{n=0}^{m} \frac{F_{n+1}^{(k)}(x)}{(1+x)^{m+k}}, \text{ by setting } p = 1/(1+x),$$

$$= 1 - \frac{F_{m+k+2}^{(k)}(x)}{x(1+x)^{m+k}}, \text{ by Lemma 3.1(b),}$$

$$\to 1 \text{ as } m \to \infty, \text{ by Lemma 3.1(a),}$$

which establishes part (b). Finally, we see that

$$P(N_k \leqslant x) = P(\emptyset) = 0, \text{ if } x < k,$$

and

$$P(N_k \leqslant x) = \sum_{n=k}^{[x]} P(N_k = n) = \sum_{n=0}^{[x]-k} P(N_k = n+k)$$

$$= \sum_{n=0}^{[x]-k} p^{n+k}F_{n+1}^{(k)}(q/p), \text{ by Theorem 3.1(a),}$$

$$= 1 - \frac{p^{[x]+1}}{q}F_{[x]+2}^{(k)}(q/p)$$

$$= 1 - \frac{p^{[x]+1}}{q} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = [x]+1}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad x \geqslant k,$$

by means of Lemma 3.1(b) and Theorem 2.1(a), both applied with $x = q/p$. The last two relations prove part (c), and this completes the proof of the theorem.

## Corollary 3.1

Let $X$ be a random variable distributed as geometric of order $k$ ($k \geqslant 1$) with parameter $p$ [6]. Then the distribution function of $X$ is given by

$$P(X \leqslant x) = \begin{cases} 1 - \dfrac{p^{[x]+1}}{q} \displaystyle\sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = [x]+1}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, & x \geqslant k, \\ 0, \text{ otherwise.} & \end{cases}$$

Proof: For $k = 1$, the definition of the geometric distribution of order $k$ implies that $X$ is distributed as geometric, so that $P(X \leqslant x) = 1 - q^{[x]}$, if $x \geqslant 1$ and 0 otherwise, which shows the corollary. For $k \geqslant 2$, the corollary is true, because of Theorem 3.1(c) and the definition of the geometric distribution of order $k$.

We end this paper by noting that Theorem 3.1(b) provides a solution to a problem proposed in [2].

## REFERENCES


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