ELEMENTAL COMPLETE COMPOSITE NUMBER GENERATORS

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Theorem

There exist arithmetic functions in closed form that are generators of all composite numbers.

Proof: It suffices to produce an example of such a function. Here, the existence of several such functions will be shown. First, consider the following two sequences:

$$2,\ \ 3, 3,\ \ 4, 4, 4,\ \ 5, 5, 5, 5,\ \ 6, 6, 6, 6, 6, \ldots\ (s_1)$$
$$2,\ \ 2, 3,\ \ 2, 3, 4,\ \ 2, 3, 4, 5,\ \ 2, 3, 4, 5, 6, \ldots\ (s_2).$$

(These sequences can be defined by specific recursions, but this will not be done here because the patterns of progression are clear.) It is easy to see that the products of corresponding terms in the sequences s_1 and s_2 constitute all the composite numbers and no prime numbers.

Second, consider the following sequence, H, which progresses

$$1,\ \ 2, 2,\ \ 3, 3, 3,\ \ 4, 4, 4, 4,\ \ 5, 5, 5, 5, 5, \ldots\ (H),$$

whose terms are those of s_1 less one. The n^{th} term of the sequence H is given by

$H(n)$ = the least integer greater than or equal to $\frac{1}{2}(\sqrt{8n+1} - 1)$.

This follows from solving for m in terms of n in the inequality

$$1 + 2 + \cdots + (m-1) < n \leqslant 1 + 2 + \cdots + m,$$

where each pair of positive integer variables m and n satisfies $H(n) = m$. Now writing

$$H(n) = \left\lceil \frac{1}{2}(\sqrt{8n+1} - 1) \right\rceil, \text{ where } \lceil x \rceil \text{ is ceiling } x,$$

it follows with little difficulty that

$$s_1(n) = H(n) + 1$$
$$s_2(n) = (n+1) - \frac{1}{2}(H(n) - 1)H(n).$$

To show the second part, one can compare the sequences

$$1,\ \ 2, 3,\ \ 4, 5, 6,\ \ 7, 8, 9, 10,\ \ 11, 12, 13, 14, 15, \ldots\ (N)$$
$$1,\ \ 1, 2,\ \ 1, 2, 3,\ \ 1, 2, 3, 4,\ \ 1, 2, 3, 4, 5, \ldots\ (I).$$

Observe that in the sequence of differences of corresponding terms

$$0,\ \ 1, 1,\ \ 3, 3, 3,\ \ 6, 6, 6, 6,\ \ 10, 10, 10, 10, 10, \ldots\ (n - I(n)),$$

the $(n+1)^{\text{th}}$ block of terms consists of the term $1 + 2 + \cdots + n$, the n^{th} triangular number, repeated a total of $n+1$ times. This implies

$$n - I(n) = \frac{1}{2}(H(n) - 1) \cdot ((H(n) - 1) + 1)$$

or

$$I(n) = n - \frac{1}{2}(H(n) - 1)H(n).$$

Multiplying together the two formulas for $s_1(n)$ and $s_2(n)$, some cancellation of product terms occurs:

$$\begin{aligned} s_1(n) \cdot s_2(n) &= (H(n) + 1) \cdot \left(n + 1 - \frac{1}{2}(H(n) - 1)H(n)\right) \\ &= (H(n) + 1)(n + 1) - \frac{1}{2}(H(n) - 1)H(n)(H(n) + 1) \\ &= nH(n) + n + H(n) + 1 - \frac{1}{2}(H^3(n) - H(n)). \end{aligned}$$

This gives a complete composite number generator

$$\mathsf{C}(n) = s_1(n) \cdot s_2(n) = (n + 1) + \left(n + \frac{3}{2}\right)H(n) - \frac{1}{2}H^3(n).$$

For comparison, a similar function which generates the positive integers—not in their natural order and with repetitions—is

$$\mathsf{N}(n) = H(n) \cdot I(n) = nH(n) + \frac{1}{2}H^2(n) - \frac{1}{2}H^3(n).$$

Alternative arithmetic generators of all the composite numbers can be found by considering sequences such as

$$2,\ 3,\ 2,\ 4,\ 3,\ 2,\ 5,\ 4,\ 3,\ 2,\ 6,\ 5,\ 4,\ 3,\ 2,\ \ldots\ (s_3)$$

$$[\text{here, } s_2(n) + s_3(n) = s_1(n) + 2]$$

and substituting $s_3(n)$ in place of either one of $s_2(n)$ or $s_1(n)$ in the product $s_1(n)s_2(n)$. Following from its relation with $s_2(n)$, an arithmetic functional form for $s_3(n)$ is found to be

$$s_3(n) = (-n + 2) + \frac{1}{2}H(n)(H(n) + 1).$$

Other complete composite number generators in closed arithmetic form are then given by

$$\overline{C}(n) = s_1(n) \cdot s_3(n) = (-n + 2) + \left(-n + \frac{5}{2}\right)H(n) + H^2(n) + \frac{1}{2}H^3(n)$$

$$\overline{\mathsf{C}}(n) = s_2(n) \cdot s_3(n) = (-n^2 + n + 2) + \frac{3}{2}H(n) + \left(n - \frac{1}{4}\right)H^2(n) - \frac{1}{4}H^4(n).$$

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