A PROPERTY OF CONVERGENTS TO THE GOLDEN MEAN

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(Submitted November 1983)

If the simple continued fraction expansion of the positive real number α is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

where a_j is a positive integer, then we denote the continued fraction expansion of α by

$$\{a_0,\ a_1,\ a_2,\ \ldots\}.$$

If

$$\beta = \{b_0,\ b_1,\ b_2,\ \ldots,\ b_{k-1},\ a_k,\ a_{k+1},\ a_{k+2},\ \ldots\},$$

then α and β are defined to be equivalent. That is, they have the same tails at some stage.

The j^{th} total convergent to α, C_j, is given by

$$C_j = \{a_0,\ a_1,\ \ldots,\ a_j\},$$

and if we represent the rational number C_j by p_j/q_j, then it can be shown that

$$p_j = p_{j-2} + a_j q_{j-1},$$
$$q_j = q_{j-2} + a_j q_{j-1}, \tag{1}$$

for $j \geqslant 0$, $p_{-2} = q_{-1} = 0$, and $q_{-2} = p_{-1} = 1$.

It is easily proved (Chrystal [1], Khintchine [2]) that

$$q_{j+1} > q_j > q_{j-1} > \cdots > q_0 = 1,$$
$$C_0 < C_2 < C_4 < \cdots < \alpha < \cdots < C_5 < C_3 < C_1,$$
$$\lim_{j\to\infty} C_j = \alpha.$$

From Le Veque [3] or Roberts [4], we have the following theorems.

Dirichlet's Theorem

If a/b is a rational fraction such that

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$$

then a/b is a total convergent to α.

Hurwitz's Theorem

If α is irrational, then there are infinitely many irreducible rational solutions a/b such that

$$\left|\alpha - \frac{a}{b}\right| < \frac{\beta}{\sqrt{5}b^2} \quad \text{for } \beta = 1.$$

In fact, if we restrict α to be an irrational which is not equivalent to $\tau = (1 + \sqrt{5})/2 = \{1, 1, 1, \ldots\}$ (the Golden Mean), then we are able to find $0 < \beta < 1$ for which there are an infinite number of solutions. For example, if α is equivalent to $\sqrt{2}$, then from Le Veque [3, p. 252] we have $\beta = \sqrt{10}/4$.

Using (1), the convergents to τ are given by

$$C_j = \frac{F_{j+1}}{F_j}, \tag{2}$$

where F_j is a term of the Fibonacci sequence $\{1, 1, 2, 3, 5, \ldots\}$ and

$$F_j = \frac{\tau^{j+1} - (1 - \tau)^{j+1}}{\sqrt{5}} \quad \text{for } j = 0, 1, 2, \ldots . \tag{3}$$

It has been shown in Roberts [4] that in the particular case where $0 < \beta < 1$ there are only finitely many irreducible rational numbers a/b such that

$$\left|\tau - \frac{a}{b}\right| < \frac{\beta}{\sqrt{5}b^2}.$$

Since $0 < \beta < 1$, then $0 < \beta/\sqrt{5} < 1/2$, and so by Dirichlet's theorem there are only finitely many total convergents to τ such that

$$|\tau - C_j| < \frac{\beta}{\sqrt{5}q_j^2}, \tag{4}$$

where C_j is given by (2).

Our purpose is to determine explicitly the finite set of convergents to τ that satisfy (4).

If j is odd ($j = 2k + 1$, $k = 0, 1, 2, \ldots$), then using (2) in (4) we seek positive values of k such that

$$|\tau - C_{2k+1}| = \frac{F_{2k+2}}{F_{2k+1}} - \tau < \frac{\beta}{\sqrt{5}F_{2k+1}^2}. \tag{5}$$

Substituting (3) in (5) and simplifying,

$$[\tau(1 - \tau)]^{2k+2} - [(1 - \tau)^2]^{2k+2} < \frac{\sqrt{5}\beta}{2\tau - 1}.$$

Using $\tau^2 = 1 + \tau$, this becomes $1 - (2 - \tau)^{2k+2} = 1 - (5 - 3\tau)^{k+1} < \beta$ or

$$\frac{1 - \beta}{5 - 3\tau} < (5 - 3\tau)^k.$$

Taking natural logarithms and using $\tau = (1 + \sqrt{5})/2$, we have

$$k > \frac{\ln\left\{\dfrac{(1 - \beta)(7 + 3\sqrt{5})}{2}\right\}}{\ln\left\{\dfrac{7 - 3\sqrt{5}}{2}\right\}}. \tag{6}$$

If j is even ($j = 2k$, $k = 0, 1, 2, \ldots$), then substituting (2) in (4) we have

$$|\tau - C_{2k}| = \tau - \frac{F_{2k+1}}{F_{2k}} < \frac{\beta}{\sqrt{5}F_{2k}^2}.$$

By reasoning similar to that which led to (6), we find that

$$k < \frac{\ln\left\{\frac{(\beta - 1)(3 + \sqrt{5})}{2}\right\}}{\ln\left\{\frac{7 - 3\sqrt{5}}{2}\right\}}. \tag{7}$$

We note that the denominator of the right-hand side of (6) is negative and so positive values of k in (6) exist only if

$$\ln\left\{\frac{(1 - \beta)(7 + 3\sqrt{5})}{2}\right\} < 0,$$

which means $1 > \beta > (3\sqrt{5} - 5)/2$.

Similarly, we see that since $0 < \beta < 1$ there are no positive values of k that satisfy (7).

Hence, there are no convergents to τ that satisfy (4) unless

$$\frac{3\sqrt{5} - 5}{2} < \beta < 1,$$

and in this case the only convergents that do satisfy (4) are given by

$$\left.\begin{aligned} &C_j = \frac{F_{j+1}}{F_j}; \quad j = 1, 3, 5, 7, \ldots, 2[R] + 1, \\ &\text{where} \\ &R = \ln\frac{(1 - \beta)(7 + 3\sqrt{5})}{2} \Big/ \ln\frac{7 - 3\sqrt{5}}{2}, \end{aligned}\right\} \tag{8}$$

and $[R]$ denotes the integer part of R. Consequently, there are $[R] + 1$ convergents to τ that satisfy (4), and these may be determined explicitly from (8).

REFERENCES

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