GOLDEN CUBOID SEQUENCES

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The notion of a golden cuboid as a generalization of a golden rectangle was first introduced by Huntley [1], where it appears as a rectangular parallelepiped with edges 1, ϕ, and ϕ^2. In this paper, we proceed from a somewhat different point of view by first recalling that a golden rectangle may be defined as the unique rectangle with the property that adjunction of a square to the larger side gives a larger rectangle geometrically similar to the first. This definition is generalized to the situation of rectangular parallelepipeds, the results being two new candidates for the title "golden cuboid" (Theorem 1). Theorem 2 establishes a nested sequence of golden cuboids analogous to the well-known sequence of nested golden rectangles. An unexpected application occurs in [2] with the construction of an interpretative model for a disputed passage of Plato's *Timaeus*, lines 31b-32c.

In searching for a generalization of the above-mentioned property of golden rectangles, let R be a rectangular parallelepiped with edges a, b, and c, and suppose $a < b < c$. A larger geometrically similar parallelepiped R' can then always be produced by the adjunction of a single parallelepiped to R, provided $b/a = c/b$. However, as a generalization of the two-dimensional case, if we in addition insist that a cube appear in the adjunction process, then it is clear that at least two adjunctions must occur. This motivates the following definition.

Definition

A rectangular parallelepiped G is golden if there is a rectangular parallelepiped G' geometrically similar to G that is obtained from G by the adjunction of two rectangular parallelepipeds, one of which is a cube.

Continuing the previous discussion, if we wish to adjoin a cube to a rectangular parallelepiped R with edges a, b, and c satisfying $a < b < c$, there must first be a prior adjunction with the effect of making two of the dimensions equal. Adjunction of a cube then retains this property, and thus the result cannot be similar to R. Consequently, we must have

$$a = b < c \quad \text{or} \quad a < b = c.$$

An elementary analysis gives the following theorem.

Theorem 1

Up to geometric similarity, there are precisely two golden cuboids, a *type one golden cuboid* with edges 1, ϕ, and ϕ and a *type two golden cuboid* with edges 1, 1, and ϕ.

Now, consider the situation where A is a type one golden cuboid with edges 1, ϕ, and ϕ, and let C be a golden cuboid of type two with edges ϕ, ϕ, and ϕ^2.

Observe that C is formed from A by the adjunction of a cube B with edge ϕ. Furthermore, if a rectangular parallelepiped D with edges 1, ϕ, and ϕ^2 is adjoined to C, we obtain a rectangular parallelepiped A' similar to A; see Figure 1. Continuing, if a cube B' with edge ϕ^2 is adjoined to A', we obtain C' similar to C. Inductively, we thus obtain the following theorem.

Theorem 2

There exists an infinite increasing sequence of nested golden cuboids,

$A_1, C_1, A_2, C_2, \ldots, A_n, C_n, \ldots,$

in which, for each n, A_n is similar to A, C_n is obtained from A_n by adjunction of a cube and is similar to C, and A_{n+1} is obtained from C_n by adjunction of a parallelepiped similar to D.

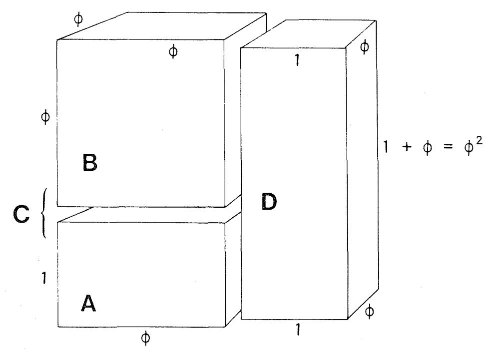

Figure 1

As in the case of golden rectangles, a decreasing sequence may be similarly constructed. We call these sequences *golden cuboid sequences.*

Remark 1: Observe that D is the "golden cuboid" of Huntley [1] but that it is neither of type one nor of type two.

Remark 2: Recall that if C is a golden rectangle and if B is a square excised by a cut parallel with the shorter side, then the remaining piece A is also golden and the areas are related by

$$\frac{\text{area}(C)}{\text{area}(B)} = \frac{\text{area}(B)}{\text{area}(A)} = \phi.$$

Now, consider the cuboid A' of the above discussion and observe that the sequence A, B, C, A' has the analogous property that

$$\frac{\text{volume}(A')}{\text{volume}(C)} = \frac{\text{volume}(C)}{\text{volume}(B)} = \frac{\text{volume}(B)}{\text{volume}(A)} = \phi.$$

REFERENCES

1. H. E. Huntley. "The Golden Cuboid." *The Fibonacci Quarterly* 2, no. 3 (1964):184.
2. M. Walker. "Mean and Extreme Ratio and Plato's *Timaeus*." (Preprint.)

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