JACOBSTHAL POLYNOMIALS AND A CONJECTURE CONCERNING FIBONACCI-LIKE MATRICES

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1. THE CONJECTURE

In the March 1983 issue of the *Mathematical Gazette* [9], Mr. Moore conjectures that if one lets

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 + x \end{bmatrix}, \tag{1.1}$$

raises Q to powers and scales each such matrix down by making the leading entry 1, then the scaled down sequence of matrices approaches

$$\begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix}, \tag{1.2}$$

as $n \to \infty$, where $\phi = (x + \sqrt{x^2 + 4})/2$.

The purpose of this paper is to show that the conjecture is true if $x > -2$, while the limit is

$$\begin{bmatrix} 1 & -\phi^{-1} \\ -\phi^{-1} & \phi^{-2} \end{bmatrix}, \tag{1.3}$$

if $x < -2$ and does not exist if $x = -2$.

It is worthwhile to mention at this point that the conjecture was first brought to the editor's attention by a letter from Mr. Moore in October 1982. The proofs of Theorems 1 to 6 were completed by Professor Bergum in November 1982. Due to the pressure of other work, the publication of these results was delayed. Several months later, the information on Jacobsthal polynomials arrived from Professor Horadam along with an alternate proof of Theorem 4. Professor Bennett joined the group by showing that (2.14) does not have a limit as n approaches infinity. The combined results are what is to follow.

If one carefully examines the way we multiply matrices, then it is quite obvious that the elements of the powers of Q satisfy linear recurrences. Examining the first five or six powers of Q, we are led to believe that

$$Q^n = \begin{bmatrix} H_n & M_n \\ M_n & N_n \end{bmatrix}, \tag{1.4}$$

where we define the sequences $\{H_n\}$, $\{M_n\}$, and $\{N_n\}$ recursively by

$$H_{n+2} = (x+2)H_{n+1} - xH_n, \quad H_1 = 1, \quad H_2 = 2, \tag{1.5}$$

$$M_{n+2} = (x+2)M_{n+1} - xM_n, \quad M_1 = 1, \quad M_2 = x+2, \tag{1.6}$$

$$N_{n+2} = (x+2)N_{n+1} - xN_n, \quad N_1 = x+1, \quad N_2 = x^2+2x+2. \tag{1.7}$$

Before proving the validity of (1.4), we first establish the following results.

Theorem 1: (a) $H_n + M_n = H_{n+1}$, (c) $(x+1)M_n + H_n = M_{n+1}$,
(b) $M_n + N_n = M_{n+1}$, (d) $(x+1)N_n + M_n = N_{n+1}$.

Proof: Since the proofs are very similar, we prove only part (c).

When $n = 1$ we have $(x+1)M_1 + H_1 = x+1+1 = x+2 = M_2$, and when $n = 2$ we have $(x+1)M_2 + H_2 = (x+1)(x+2) + 2 = x^2 + 3x + 4 = M_3$; so that (c) is true for $n = 1$ and 2. Now assume the statement is true for all positive integers less than k where $k \geqslant 3$. Then by (1.6), (1.5), and the induction hypothesis, we have

$$\begin{aligned}(x+1)M_k + H_k &= (x+1)[(x+2)M_{k-1} - xM_{k-2}] + [(x+2)H_{k-1} - xH_{k-2}] \\ &= (x+2)[(x+1)M_{k-1} + H_{k-1}] - x[(x+1)M_{k-2} + H_{k-2}] \\ &= (x+2)M_k - xM_{k-1} = M_{k+1},\end{aligned}$$

and (c) is proved.

The proof of (1.4) follows directly from Theorem 1 by mathematical induction giving

Theorem 2: If $Q = \begin{bmatrix} 1 & 1 \\ 1 & 1+x \end{bmatrix}$ then $Q^n = \begin{bmatrix} H_n & M_n \\ M_n & N_n \end{bmatrix}$ for all integers $n \geqslant 1$.

Now we scale down Q^n and obtain a new sequence of matrices $\{R_n\}$ where

$$R_n = \begin{bmatrix} 1 & M_n/H_n \\ M_n/H_n & N_n/H_n \end{bmatrix}, \tag{1.8}$$

and then ask: What happens as $n \to \infty$? To answer this question, we first apply (1.6) and (1.7) found in [6] and obtain

$$H_n = \frac{(2-\beta)\alpha^{n-1} - (2-\alpha)\beta^{n-1}}{\alpha - \beta}, \quad n \geqslant 1, \tag{1.9}$$

$$M_n = \frac{(x + 2 - \beta)\alpha^{n-1} - (x + 2 - \alpha)\beta^{n-1}}{\alpha - \beta}, \quad n \geqslant 1, \tag{1.10}$$

$$N_n = \frac{[x^2 + 2x + 2 - (x + 1)\beta]\alpha^{n-1} - [x^2 + 2x + 2 - (x + 1)\alpha]\beta^{n-1}}{\alpha - \beta}, \quad n \geqslant 1, \tag{1.11}$$

where

$$\alpha = \frac{(x + 2) + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{(x + 2) - \sqrt{x^2 + 4}}{2}. \tag{1.12}$$

are the roots of the characteristic equation arising from the recurrences (1.5), (1.6), and (1.7). Next, we analyze the range of β/α and α/β, as this is needed before we can find

$$\lim_{n \to \infty} \frac{M_n}{H_n} \quad \text{and} \quad \lim_{n \to \infty} \frac{N_n}{H_n}.$$

If $x > -2$, then $0 < 2(x^2 + 4) + 2(x + 2)\sqrt{x^2 + 4}$, so that

$$4x < [x^2 + 4x + 4 + 2(x + 2)\sqrt{x^2 + 4} + x^2 + 4] = [(x + 2) + \sqrt{x^2 + 4}]^2$$

or

$$1 > 4x/[(x + 2) + \sqrt{x^2 + 4}]^2.$$

When $x \neq 0$ we can multiply and divide the right side of the last inequality by $(x + 2 - \sqrt{x^2 + 4})$ to obtain

$$1 > \frac{x + 2 - \sqrt{x^2 + 4}}{x + 2 + \sqrt{x^2 + 4}} = \frac{\beta}{\alpha}.$$

If $x = 0$, then $\beta = 0$ and $\alpha = 2$, so that $\beta/\alpha = 0 < 1$. Since $x > -2$, we also have

$$0 < x + 2 + \sqrt{x^2 + 4} \quad \text{or} \quad 0 < 2(x + 2)^2 + 2(x + 2)\sqrt{x^2 + 4},$$

so that

$$-4x < 2x^2 + 4x + 8 + 2(x + 2)\sqrt{x^2 + 4}.$$

Hence,

$$4x > -[(x + 2) + \sqrt{x^2 + 4})^2 \quad \text{or} \quad -1 < 4x/[(x + 2) + \sqrt{x^2 + 4}]^2.$$

Operating as before when $x \neq 0$, we see that

$$-1 < \frac{(x + 2) - \sqrt{x^2 + 4}}{(x + 2) + \sqrt{x^2 + 4}} = \frac{\beta}{\alpha},$$

which is also true if $x = 0$. Therefore,

$$-1 < \frac{\beta}{\alpha} < 1, \quad \text{if } x > -2. \tag{1.13}$$

When $x < -2$, we have

$$x + 2 < \sqrt{x^2 + 4} \quad \text{or} \quad 2(x + 2)^2 > 2(x + 2)\sqrt{x^2 + 4},$$

so that

$$2x^2 + 4x + 8 - 2(x + 2)\sqrt{x^2 + 4} = (x + 2 - \sqrt{x^2 + 4})^2 > -4x.$$

Hence,

$$-1 < 4x/(x + 2 - \sqrt{x^2 + 4})^2 = \frac{x + 2 + \sqrt{x^2 + 4}}{x + 2 - \sqrt{x^2 + 4}} = \frac{\alpha}{\beta}.$$

Since $x < 0$,

$$x^2 + 4x + 4 < x^2 + 4 \quad \text{or} \quad \sqrt{(x + 2)^2} < \sqrt{x^2 + 4}.$$

Therefore,

$$|x + 2| < \sqrt{x^2 + 4} \quad \text{and} \quad x + 2 > -\sqrt{x^2 + 4},$$

so that $\alpha > 0$. However, $\beta < 0$ and we get

$$-1 < \frac{\alpha}{\beta} < 0, \quad \text{if } x < -2. \tag{1.14}$$

When $x = -2$, we have $\alpha/\beta = \beta/\alpha = -1$. Combining these results, we obtain

<u>Theorem 3</u>: If $\alpha = \dfrac{x + 2 + \sqrt{x^2 + 4}}{2}$ and $\beta = \dfrac{x + 2 - \sqrt{x^2 + 4}}{2}$, then

(a) $\dfrac{\alpha}{\beta} = \dfrac{\beta}{\alpha} = -1$, if $x = -2$,

(b) $-1 < \dfrac{\alpha}{\beta} < 0$, if $x < -2$,

(c) $-1 < \dfrac{\beta}{\alpha} < 1$, if $x > -2$.

Let $x > -2$ and $x \neq 0$; then by Theorem 3(c), substitution of β, and rationalization

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \frac{x + 2 - \beta}{2 - \beta} = \frac{x + 2 + \sqrt{x^2 + 4}}{(2 - x) + \sqrt{x^2 + 4}} = \frac{x + \sqrt{x^2 + 4}}{2} = \phi.$$

Also, using similar steps, we have

$$\lim_{n\to\infty} \frac{N_n}{H_n} = \frac{x^2 + 2x + 2 - (x + 1)\beta}{2 - \beta} = \frac{x^2 + x + 2 + (x + 1)\sqrt{x^2 + 4}}{(2 - x) + \sqrt{x^2 + 4}}$$

$$= \frac{2x^2 + 4 + 2x\sqrt{x^2 + 4}}{4} = \phi^2.$$

If $x = 0$, then $M_n/H_n = N_n/H_n = 1$ for all n, so that

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \lim_{n\to\infty} \frac{N_n}{H_n} = 1.$$

Hence, we have

<u>Theorem 4</u>: If $x > -2$, then $\lim\limits_{n\to\infty} R_n = \begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix}$.

Let us now assume that $x < -2$; then reasoning as above, we have

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \frac{x + 2 - \alpha}{2 - \alpha} = \frac{x + 2 - \sqrt{x^2 + 4}}{2 - x - \sqrt{x^2 + 4}} = -\frac{2}{x + \sqrt{x^2 + 4}} = -\phi^{-1}.$$

Similarly,

$$\lim_{n\to\infty} \frac{N_n}{H_n} = \frac{x^2 + 2x + 2 - (x+1)\alpha}{2 - \alpha} = \phi^{-2},$$

and we have in (1.8)

Theorem 5: If $x < -2$, then $\lim_{n\to\infty} R_n = \begin{bmatrix} 1 & -\phi^{-1} \\ -\phi^{-1} & \phi^{-2} \end{bmatrix}$.

When $x = -2$, $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in (1.1) so that $R_n = \begin{cases} Q, & \text{if } n \text{ is odd} \\ I, & \text{if } n \text{ is even} \end{cases}$, where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, we obtain in (1.8)

Theorem 6: If $x = -2$, then $\lim_{n\to\infty} R_n$ does not exist.

Observe that when $x = -1$, (1.5), (1.6), and (1.7) all reduce to the definition for the sequence of Fibonacci numbers and (1.1) becomes

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

which is discussed in [1], [2], [3], and [4].

2. JACOBSTHAL POLYNOMIALS AND MATRICES

The *Jacobsthal polynomials* $J_n(x) \equiv J_n$ are defined in [7] by the recurrence relation

$$J_{n+2} = J_{n+1} + xJ_n \qquad (J_0 = 0,\ J_1 = 1) \tag{2.1}$$

and the first few term of $\{J_n\}$ are

$$\begin{array}{ccccccc} J_1 & J_2 & J_3 & J_4 & J_5 & J_6 & \cdots \\ 1 & 1 & 1+x & 1+2x & 1+3x+x^2 & 1+4x+3x^2 & \cdots \end{array} \tag{2.2}$$

The matrix (1.1) can now be expressed as

$$J = \begin{bmatrix} J_1 & J_2 \\ J_2 & J_3 \end{bmatrix}, \tag{2.3}$$

and justifiably called a *Jacobsthal matrix*.

Powers of this matrix obviously do not have Jacobsthal polynomials as their entries.

Therefore, two questions arise:

(i) How may the Jacobsthal matrices $\begin{bmatrix} J_n & J_{n+1} \\ J_{n+1} & J_{n+2} \end{bmatrix}$, $n > 2$, be generated?

(ii) What is the result if we scale these matrices down as in (1.8) and let $n \to \infty$?

The answer to (i) is associated with the matrix H [$\equiv H(x)$]

$$H = \begin{bmatrix} 0 & 1 \\ x & 1 \end{bmatrix}. \tag{2.4}$$

Using (2.1)-(2.4) and induction, we readily obtain

$$H^n J = \begin{bmatrix} J_{n+1} & J_{n+2} \\ J_{n+2} & J_{n+3} \end{bmatrix}, \tag{2.5}$$

so question (i) is answered.

Let the matrices generated by powers of H in (2.5) be represented as

$$\mathscr{J}_n = H^n J. \tag{2.6}$$

We call the set of matrices $\{\mathscr{J}_n\}$ the *Jacobsthal matrices*, since all their entries are Jacobsthal polynomials.

Scaling down the Jacobsthal matrices, we have

$$\mathscr{J}_n^* = \begin{bmatrix} 1 & \dfrac{J_{n+2}}{J_{n+1}} \\ \dfrac{J_{n+2}}{J_{n+1}} & \dfrac{J_{n+3}}{J_{n+1}} \end{bmatrix}. \tag{2.7}$$

Now, the Binet form for J_n can be found by routine measures (see [2] and [8]) to be

$$J_n = \frac{\gamma^n - \delta^n}{\sqrt{1 + 4x}}, \quad x \neq -\frac{1}{4}, \tag{2.8}$$

where

$$\gamma = \frac{1 + \sqrt{1 + 4x}}{2}, \quad \delta = \frac{1 - \sqrt{1 + 4x}}{2} \tag{2.9}$$

are the roots of the characteristic equation

$$\lambda^2 - \lambda - x = 0 \tag{2.10}$$

for the recurrence relation (2.1).

Let $x > -1/4$. Elementary calculations reveal that $|\delta/\gamma| < 1$. Hence,

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \gamma, \tag{2.11}$$

and

$$\lim_{n\to\infty} \frac{J_{n+2}}{J_n} = \gamma^2, \tag{2.12}$$

so that the limiting form of $\mathcal{J}_n^*$ is

$$\begin{bmatrix} 1 & \gamma \\ \gamma & \gamma^2 \end{bmatrix} \tag{2.13}$$

When $x = -1/4$, $\gamma = \delta = 1/2$. Hence, $J_n = n/2^{n-1}$ by standard methods of difference equations where the roots of the characteristic equation are equal. Therefore, (2.13) still holds.

If $x < -1/4$, then from (2.1)

$$J_n = \frac{2(\sqrt{-x})^n}{\sqrt{-1-4x}} \sin(n\tau)$$

where $\cos\tau = 1/2\sqrt{-x}$ and $\sin\tau = \sqrt{-1-4x}/2\sqrt{-x}$. Therefore,

$$\frac{J_{n+1}}{J_n} = \sqrt{-x}\,\frac{\sin(n+1)\tau}{\sin(n\tau)} = \left(\frac{1}{2} + \frac{(\cot n\tau)\sqrt{-1-4x}}{2}\right). \tag{2.14}$$

<u>Theorem 7</u>: There is no real number τ having the property that

$\lim\limits_{n\to\infty} \cot(n\tau)$ exists as a finite real number or $\pm\infty$.

<u>Case I</u>. Suppose that τ is a rational multiple of π, say $\tau = (p/q)\pi$, where p is an integer and q is a natural number. Then $\cot(n\tau)$ is not even defined for integers n that are multiples of q.

In each of the cases to follow, it will be assumed that τ is *not* a rational multiple of π. Then $\sin\tau \neq 0$ and $\sin(n\tau) \neq 0$ for any positive integer n. So the formula

$$\cot(n+1)\tau = \frac{\cot(n\tau)\cot\tau - 1}{\cot(n\tau) + \cot\tau} \tag{2.15}$$

is valid. Note also that $\cot\tau \neq 0$. Furthermore, $\cot(n\tau) \neq 0$ for any positive integer n since this would imply that τ is a rational multiple of π.

<u>Case II</u>. If $\lim \cot(n\tau) = \pm\infty$, then (2.15) yields

$$\infty = \lim_{n\to\infty} \cot(n\tau) = \lim_{n\to\infty} \cot(n+1)\tau = \lim_{n\to\infty} \frac{\cot\tau - \dfrac{1}{\cot(n\tau)}}{1 + \dfrac{\cot\tau}{\cot(n\tau)}} = \cot\tau,$$

which is impossible.

<u>Case III</u>. Suppose that $\lim\limits_{n\to\infty} \cot(n\tau) = r$, where r is some real number. Set $s = \cot\tau$. If $r + s \neq 0$, then from (2.15),

$$r = \frac{rs - 1}{r + s},$$

$$r^2 + rs = rs - 1,$$

and

$$r^2 = -1, \text{ which is impossible.}$$

If $r + s = 0$, then in order to obtain a finite limit in (2.15), it must follow that $rs - 1 = 0$. Thus,

$$r = -s = \frac{1}{s}$$

or

$$s^2 = -1, \text{ which is impossible.}$$

It has now been shown that, for all possible choices of τ, $\lim_{n\to\infty} \cot(n\tau)$ cannot exist. Hence, $\operatorname{Lim}_{n\to\infty}(J_{n+1}/J_n)$ does not exist.

Much more can be said about other properties of the Jacobsthal polynomials J_n. They are, in fact, a special case of the $w_n(a, b; p, q)$ discussed in [6], where $p = 1$, $q = -x$. See the Historical Note below for Jacobsthal's original contributions and [5] for additional properties.

3. HISTORICAL NOTE

The recurrence relation (2.1) is associated with the name of Jacobsthal [7] who, in 1919, seems to be the first to record it. His notation is related to ours by the correspondence where $F_n(x)$ are the Fibonacci polynomials defined by $F_1(x) = 1$, $F_2(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F(x)$.

Using methods different from ours, Jacobsthal established the Binet form (2.8). Among other basic results demonstrated by him are, in his notation,

(a) the explicit summation formula

$$F_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k$$

and

(b) the extension of the definition of $F_n(x)$ to negative values of n. That is,

$$F_{-n}(x) = (-1)^n \frac{F_{n-2}(x)}{x^{n-1}}, \quad n \geqslant 1.$$

Both of the above results can be readily converted, with due care, into our J-notation by means of the stated correspondence.

Although Jacobsthal alludes to the polynomials (2.2) as "Fibonacci polynomials," they are now known by his name; in fairness, then, the matrices whose entries are Jacobsthal polynomials must also bear his name.

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