A THIRD-ORDER ANALOG OF A RESULT OF L. CARLITZ

VICHIAN LAOHAKOSOL and NIT ROENROM
The University of Texas at Austin, Austin, TX 78712

(Submitted March 1983)

1. INTRODUCTION

In 1966, L. Carlitz [1] employed a technique based on a generating function to solve completely the second-order difference equation

$$f_{n+2}(x) = (x + 2n + p + 1)f_{n+1}(x) - (n^2 + pn + q)f_n(x), \quad (n = 0, 1, 2, \ldots),$$

with the initial conditions

$$f_0(x) = 0, \; f_1(x) = 1,$$

and p, q are parameters subject only to the restriction

$$p^2 - 4q \neq 0.$$

The polynomials $f_n(x)$ are known to be orthogonal on the real line with respect to some weight function.

Though the difference equation considered by Carlitz is of a special form, by studying Carlitz's proof, it is evident that his technique can also be used to solve analogous difference equations of higher order. It is our purpose here to illustrate this by way of solving completely the following third-order difference equation:

$$\begin{aligned} f_{n+3}(x) = (x^2 + 3pn + q)f_{n+2}(x) &+ \{-3p^2n^2 + (3p^2 - 2pq)n + r\}f_{n+1}(x) \\ &+ \{p^3n^3 + (-3p^2 + p^2q)n^2 + (2p^3 - p^2q - pr)n + s\}f_n(x), \\ &(n = 0, 1, 2, \ldots), \quad (1) \end{aligned}$$

with the initial conditions

$$f_0(x) = f_1(x) = 0, \; f_2(x) = 1, \tag{2}$$

and p, q, r, s are arbitrary parameters subject to the following three restrictions:

I. $p \neq 0$,

II. all three roots λ_1, λ_2, λ_3 of the equation
$$p^3\lambda^3 + (3p^3 - p^2q)\lambda^2 + (2p^3 - p^2q - pr)\lambda - s = 0$$
are distinct and none is a nonpositive integer,

III. both roots μ_1 and μ_2 of the equation
$$p^3\mu^2 + (3\lambda p^3 + 3p^3 - p^2q)\mu + (3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2q - pr = 0,$$
where λ denotes any one of λ_1, λ_2, or λ_3 from II, are nonpositive integers.

2. THE METHOD

Let

$$F(t) := F(t, x) = \sum_{n=0}^{\infty} f_n(x)\frac{t^n}{n!} \tag{3}$$

be a generating function for $f_n(x)$. From (1), (2), and (3) we get

$$(1 - pt)^3 F'''(t) - q(1 - pt)^2 F''(t) - r(1 - pt)F'(t) - sF(t) = x^2 F''(t).$$

We remark here that, save the right-hand side, this differential equation resembles the well-known Euler linear differential equation (see, e.g., Ince [2], pp. 141-143).

Next, we define an operator

$$\Delta := (1 - pt)^3 D^3 - q(1 - pt)^2 D^2 - r(1 - pt)D - s, \quad (D = d/dt).$$

Then our differential equation becomes

$$\Delta F(t) = x^2 F''(t).$$

We expect three independent solutions of this differential equation to be of the form

$$\phi(t, \lambda) := \phi(t, \lambda, x) = \sum_{k=0}^{\infty} T_k x^k (1 - pt)^{-\lambda-k},$$

where λ is any one of λ_1, λ_2, λ_3. Thus, we must compute $T_k = T_k(\lambda)$.

By direct computation, we get

$$\frac{\Delta(1 - pt)^{-\lambda-k}}{(1 - pt)^{-\lambda-k}} = (\lambda + k)(\lambda + k + 1)(\lambda + k + 2)p^3 - (\lambda + k)(\lambda + k + 1)p^2 q - (\lambda + k)pr - s.$$

Equating the coefficients of $x^k(1 - pt)^{-\lambda-k}$ for $k \geqslant 2$ in

$$\Delta\phi(t, \lambda) = x^2\phi''(t, \lambda), \tag{4}$$

we get

$$T_k = \frac{(\lambda + k - 2)(\lambda + k + 1)p^2}{(\lambda + k)(\lambda + k + 1)(\lambda + k + 2)p^3 - (\lambda + k)(\lambda + k + 1)p^2 q - (\lambda + k)pr - s} T_{k-2}.$$

Making use of restriction II that λ is a (nonpositive integer) root of

$$p^3\lambda^3 + (3p^3 - p^2 q)\lambda^2 + (2p^3 - p^2 q - pr)\lambda - s = 0,$$

we have

$$T_k = \frac{(\lambda + k - 2)(\lambda + k - 1)p^2}{k[p^3 k^2 + (3p^3 \;\; + 3p^3 - p^2 q)k + \{(3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2 q - pr\}]} T_{k-2}.$$

Also, making use of condition III that both roots μ of

$$p^3\mu^2 + (3p^3\lambda + 3p^3 - p^2 q)\mu + \{(3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2 q - pr\} = 0$$

are nonpositive integers, we arrive at the fact that

$$T_k = \frac{(\lambda + k - 2)(\lambda + k - 1)}{k(k - \mu_1)(k - \mu_2)p} T_{k-2}$$

is well defined. Consequently,

$$T_{2k} = T_0 p^{-k} \prod_{\ell=1}^{k} \frac{(2\ell - 2 + \lambda)(2\ell - 1 + \lambda)}{2\ell(2\ell - \mu_1)(2\ell - \mu_2)} = \frac{\left(\frac{\lambda}{2}\right)_k \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k}{p^k 2^k k! \left(1 - \frac{\mu_1}{2}\right)_k \left(1 - \frac{\mu_2}{2}\right)_k} T_0,$$

where $(y)_k = y(y + 1) \cdots (y + k - 1)$, and

$$T_{2k+1} = \frac{2^k k! \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k \left(\frac{\lambda}{2} + 1\right)_k}{p^k (2k + 1)! \left(\frac{3}{2} - \frac{\mu_1}{2}\right)_k \left(\frac{3}{2} - \frac{\mu_2}{2}\right)_k} T_1.$$

Thus,

$$\phi(t, \lambda) = \sum_{k=0}^{\infty} \{T_{2k} x^{2k} (1 - pt)^{-\lambda-2k} + T_{2k+1} x^{2k+1} (1 - pt)^{-\lambda-2k-1}\}.$$

Since the degree (in x) of $f_n(x)$ is even, we must choose $T_1 = 0$. Also, we have to adjust the initial conditions; equating the coefficients of $x^0(1 - pt)^{-\lambda-0}$ in (4) and using restriction II, we may take $T_0 = 1$. Thus,

$$\phi(t, \lambda) = \sum_{k=0}^{\infty} T_{2k} x^{2k} (1 - pt)^{-\lambda-2k} = \sum_{k=0}^{\infty} T_{2k} x^{2k} \sum_{n=0}^{\infty} (\lambda + 2k)_n p^n \frac{t^n}{n!},$$

where

$$T_{2k} = \frac{\left(\frac{\lambda}{2}\right)_k \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k}{p^k 2^k k! \left(1 - \frac{\mu_1}{2}\right)_k \left(1 - \frac{\mu_2}{2}\right)_k}, \quad (k = 0, 1, 2, \ldots).$$

Let $c_n(\lambda) := c_n(\lambda, x)$ be the coefficient of $t^n/n!$ in $\phi(t, \lambda)$. Then

$$c_n(\lambda) = \sum_{k=0}^{\infty} T_{2k} (\lambda + 2k)_n p^n x^{2k}.$$

Hence, we have the general solution to (1) as

$$f_n(x) = w_1 c_n(x, \lambda_1) + w_2 c_n(x, \lambda_2) + w_3 c_n(x, \lambda_3),$$

where

$$w_i = w_i(x, \lambda_1, \lambda_2, \lambda_3), \quad (i = 1, 2, 3)$$

are to be chosen so that the initial conditions (2) are fulfilled, namely:

$$0 = w_1 c_0(\lambda_1) + w_2 c_0(\lambda_2) + w_3 c_0(\lambda_3);$$
$$0 = w_1 c_1(\lambda_1) + w_2 c_1(\lambda_2) + w_3 c_1(\lambda_3);$$
$$1 = w_1 c_2(\lambda_1) + w_2 c_2(\lambda_2) + w_3 c_2(\lambda_3).$$

Solving this system of equations, we get

$$Dw_1 = c_0(\lambda_2)c_1(\lambda_3) - c_0(\lambda_3)c_1(\lambda_2)$$
$$Dw_2 = c_0(\lambda_3)c_1(\lambda_1) - c_0(\lambda_1)c_1(\lambda_3),$$
$$Dw_3 = c_0(\lambda_1)c_1(\lambda_2) - c_0(\lambda_2)c_1(\lambda_1),$$

where

$$D := D(x, \lambda_1, \lambda_2, \lambda_3)$$
$$= \det \begin{bmatrix} c_0(\lambda_1) & c_0(\lambda_2) & c_0(\lambda_3) \\ c_1(\lambda_1) & c_1(\lambda_2) & c_1(\lambda_3) \\ c_2(\lambda_1) & c_2(\lambda_2) & c_2(\lambda_3) \end{bmatrix}.$$

It can be verified that $D \not\equiv 0$. With these values, we have completely solved our difference equation.

3. AN EXAMPLE

In closing, we give a more specific example to our result. Take $p = 1$, $q = 4$, $r = -3$, $s = 1$. The difference equation (1) then becomes

$$f_{n+3}(x) = (x^2 + 3n + 4)f_{n+2}(x) + (-3n^2 - 5n - 3)f_{n+1}(x) + (n^3 + n^2 + n + 1)f_n(x).$$

The three roots of

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

are

$$\lambda_1 = 1,\ \lambda_2 = i = \sqrt{-1},\ \lambda_3 = -i.$$

The roots of

$$\mu^2 + (3\lambda - 1)\mu + (3\lambda^2 - 2\lambda + 1) = 0$$

for the corresponding λ are

$$\lambda_1 = 1 : \mu_{11} = \sqrt{2}\exp\left(\frac{3\pi i}{4}\right),\ \mu_{12} = \sqrt{2}\exp\left(\frac{5\pi i}{4}\right),$$

$$\lambda_2 = i : \mu_{21} = \sqrt{2}\exp\left(\frac{7\pi i}{4}\right),\ \mu_{22} = \sqrt{2}\exp\left(\frac{3\pi i}{2}\right),$$

$$\lambda_3 = -i : \mu_{31} = \sqrt{2}\exp\left(\frac{\pi i}{4}\right),\ \mu_{32} = \sqrt{2}\exp\left(\frac{\pi i}{2}\right).$$

Also,

$$T_{2k}(\lambda_1) = T_{2k}(1) = \frac{(2k)!}{2^k k! \prod_{j=1}^{k}[(2j+1)^2 + 1]},\ (k = 0, 1, 2, \ldots),$$

$$T_{2k}(\lambda_2) = T_{2k}(i) = \frac{(i/2)_k}{k!2^k(1+i)_k},$$

$$T_{2k}(\lambda_3) = T_{2k}(-i) = \frac{(-i/2)_k}{k!2^k(1-i)_k},$$

$$c_n(\lambda_1) = c_n(1) = \sum_{k=0}^{\infty} \frac{(2k+n)!x^k}{2^k k! \prod_{j=1}^{k}[(2j+1)^2 + 1]},\ (n = 0, 1, 2, \ldots),$$

$$c_n(\lambda_2) = c_n(i) = \sum_{k=0}^{\infty} \frac{(i/2)_k (i + 2k)_n}{k! 2^k (1 + i)_k} x^{2k},$$

$$c_n(\lambda_3) = c_n(-i) = \sum_{k=0}^{\infty} \frac{(-i/2)_k (-i + 2k)_n}{k! 2^k (1 - i)_k} x^{2k}.$$

If we consider the case where $x = 0$, then we get

$$c_n(\lambda_1, 0) = n!, \; c_n(\lambda_2, 0) = (i)_n,$$
$$c_n(\lambda_3, 0) = (-i)_n, \; (n = 0, 1, 2, \ldots),$$

and

$$w = \frac{1}{2}, \; w = \frac{1}{4}(-1 + i), \; w = \frac{1}{4}(-1 - i).$$

Hence,

$$f_n(0) = \frac{1}{2}n! + \frac{1}{4}(-1 + i)(i)_n + \frac{1}{4}(-1 - i)(-i)_n.$$

This solution can be directly checked via the differential equation

$$(1 - t)^3 F'''(t) - 4(1 - t)^2 F''(t) + 3(1 - t)F'(t) - F(t) = 0,$$

which is the familiar Euler linear differential equation.

The solution with initial conditions

$$f(0) = F(0) = 0, \; f_1(0) = F'(0) = 0, \; f_2(0) = F''(0) = 1$$

is given by (see, e.g., Ince [2], pp. 140-141)

$$F(t) = \frac{1}{2}(1 - t)^{-1} + \frac{1}{4}(-1 + i)(1 - t)^{-i} + \frac{1}{4}(-1 - i(1 - t))^{i}$$

and it can be immediately verified that this agrees with the solution found above.

REFERENCES

1. L. Carlitz. "Some Orthogonal Polynomials Related to Fibonacci Numbers." *The Fibonacci Quarterly 4*, no. 1 (1966):43-48.

2. E. L. Ince. *Ordinary Differential Equations*. New York: Dover, 1956.

◆◇◆◇◆