ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-392 *Proposed by Piero Filipponi, Rome, Italy*

It is known [1], [2], [3], [4] that every positive integer n can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). Denoting by $f(n)$ the number of F-addends the sum of which represents the integer n and denoting by $[x]$ the greatest integer not exceeding x, prove that:

(i) $f([F_k/2]) = [k/3], \quad (k = 3, 4, \ldots);$

(ii) $$f([F_k/3]) = \begin{cases} [k/4] + 1, & \text{for } [k/4] \equiv 1 \pmod 2 \text{ and } k \equiv 3 \pmod 4 \\ [k/4], & \text{otherwise.} \end{cases} \quad (k = 4, 5, \ldots)$$

Find (if any) a closed expression for $f(F_k/p)$ with p a prime and k such that $F_k \equiv 0 \pmod p$.

References

1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly 2*, no. 4 (1964):163-168.
2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly 3*, no. 1 (1965):1-8.
3. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers." *J. London Math. Soc. 35* (1960):143-160.
4. D. A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly 6*, no. 4 (1968):235-244.

H-393 *Proposed by M. Wachtel, Zürich, Switzerland*

Consider the triangle below:

A_{-n}		...		A_{-4}	A_{-3}	A_{-2}	A_{-1}	$A_0 = m^2$	A_1	A_2	A_3	A_4	...		A_n
							-1	1							
						-1	5	9	11						
					1	11	19	25	29	31					
				5	19	31	41	49	55	59	61				
			11	29	45	59	71	81	89	95	99	101			
		19	41	61	79	95	109	121	131	139	145	149	151		
	29	55	79	101	121	139	155	169	181	191	199	205	209	211	
41	71	99	125	149	171	191	209	225	239	251	261	269	275	279	281
			...					...				...			

This triangle shows two types of sequences:

a) with primes, or with composite terms with no divisors congruent to 3 or 7 modulo 10;

b) as described in a), and, in addition, terms with divisors congruent to (3 or 7 modulo $10)^{2k}$.

In the above triangle, let:

$A_{0,m} = m^2$ (m odd)

$A_{-n,m}$ = the terms on the left of $A_{0,m}$

$A_{n,m}$ = the terms on the right of $A_{0,m}$

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1. Establish general formulas for the sequences of every row, every column, and every diagonal.

2. Establish formulas:

   a) for the sequences showing terms that either are primes, or else composite integers with no divisors congruent to 3 or 7 modulo 10;

   b) for the sequences with terms as described in a) and also with composite terms showing periodically also divisors congruent to

   (3 or 7 modulo $10)^{2k}$.

Remarks: Apart from the formulas

$$C - N^2 + rN$$

for the finite sequences, and

$$C + mN^2 + rN$$

for the infinite sequences, there exist other construction rules.

Some examples of relationships which can easily be established are:

Column A_{-2} Down	Diag. A_2 Down Left	Columns A_{-2} and A_2 Down
$-1 = -1 \cdot 2 + 1$	$31 = -2 \cdot 9 + 7$	Every term plus $5 = (m \mp 1)^2$
$11 = 1 \cdot 2 + 3$	$55 = -1 \cdot 9 + 8$	Columns A_{-4} and A_4 Down
$31 = 3 \cdot 2 + 5$	$81 = 0 \cdot 9 + 9$	
$59 = 5 \cdot 2 + 7$	$109 = 1 \cdot 9 + 10$	Every term plus $5 \cdot 2^2 = (m \mp 2)^2$
$\vdots$	$\vdots$	

According to what is stated in 2 above, the following rule holds true:

> $A_{n,m} + 5(a^2 + a) + 1 = b^2 + b$, with infinitely many solutions whereby a and b are F/L numbers.

Example: $A_{2,11}$ and $A_{-2,13} = 139$

	a	b
I.	$5F_{2+6n} - \dfrac{F_{-1+6n} + 1}{2}$	$5L_{2+6n} - \dfrac{L_{-1+6n} + 1}{2}$
II.	$F_{6+6n} + \dfrac{L_{-1+6n} - 1}{2}$	$10F_{3+6n} - \dfrac{L_{-1+6n} + 1}{2}$
III.	$5F_{4+6n} - \dfrac{F_{1+6n} + 1}{2}$	$5L_{4+6n} - \dfrac{L_{1+6n} + 1}{2}$
IV.	$F_{8+6n} + \dfrac{L_{1+6n} - 1}{2}$	$10F_{5+6n} - \dfrac{L_{1+6n} + 1}{2}$

Special Properties: All sequences emerging out of this triangle show the following property:

> $A_k \cdot A_{k+d} + B$ yield either a square or, alternately, a product of two consecutive integers. For brevity, example and formula are omitted.

Then there are combinations of different sequences, but it would take too much space to pursue the many things involved in this triangle.

SOLUTIONS

AB Surd

H-367 *Proposed by M. Wachtel, Zürich, Switzerland*
(Vol. 22, no. 1, February 1984)

Problem A

Prove the identity:

$$\sqrt{(L_{2n} - L_{n-2}^2) \cdot (L_{2n+4} - L_n^2) + 30} = 5F_{2n} - 3(-1)^n$$

Problem B

Prove the identities:

$$\left.\begin{aligned}&\sqrt{(F_{n+1}^2 - F_{2n+3})\cdot(F_{n+3}^2 - F_{2n+7})}\\&\sqrt{(F_{n+3}^2 - F_{2n+5})\cdot(F_{n+5}^2 - F_{2n+9})}\\&\sqrt{(F_{n+4}^2 - F_{2n+6})\cdot(F_{n+6}^2 - F_{2n+10})}\end{aligned}\right\} = F_{n+2}F_{n+4} \quad\text{or}\quad F_{n+3}^2 + (-1)^n$$

Solution by the proposer.

1. These are particular instances of the more general identities:

A) $$\sqrt{(L_{2n+m} - L_{n-2+m}^2)\cdot(L_{2n+4+m} - L_{n+m}^2) + \overbrace{5[L_{4-m} - (-1)^m]}^{A'}}$$
$$= L_{2n+2+m} - L_{n-2+m}L_{n+m}$$

B) $$\sqrt{(F_{2n+m} - F_{n-2+m}^2)\cdot(F_{2n+4+m} - F_{n+m}^2) - \overbrace{(-1)^m(F_{m-4} - 1)}^{B'}}$$
$$= F_{2n+2+m} - F_{n-2+m}F_{n+m}$$

$n, m = 0 \pm 1, 2, 3, \ldots$.

2. Squaring both sides of 1 and making use of the following identities on the left-hand side, we obtain (with these identities the congruence of both sides is established):

A) I. $L_{2n+m}L_{2n+4+m} \equiv L_{2n+2+m}^2 + 5(-1)^m$

(derived from $L_{2n}L_{2n+4} \equiv L_{2n+2}^2 + 5$)

II. $L_{2n+m}L_{n+m}^2 \equiv L_{2n+2+m}L_{n-2+m}L_{n+m} - 5(-1)^{n+m}L_{n+m}F_{n+2}$

[derived from $L_{2n}L_n \equiv L_{2n+2}L_{n-2} - 5(-1)^nF_{n+2}$
and $L_{2n+m}L_{n+m} \equiv L_{2n+2+m}L_{n-2+m} - 5(-1)^{n+m}F_{n+2}$]

III. $L_{2n+4+m}L_{n-2+m}^2 - 5L_{4-m} \equiv L_{2n+2+m}L_{n-2+m}L_{n+m} + 5(-1)^{n+m}L_{n+m}F_{n+2}$

(similarly derived as I and II)

B) I. $F_{2n+m}F_{2n+4+m} \equiv F_{2n+2+m}^2 - (-1)^m$

II. $F_{2n+m}F_{n+m}^2 \equiv F_{2n+2+m}F_{n-2+m}F_{n+m} + (-1)^{n+m}F_{n+m}F_{n+2}$

III. $F_{2n+4+m}F_{n-2+m}^2 + (-1)^mF_{m-4} \equiv F_{2n+2+m}F_{n-2+m}F_{n+m} - (-1)^{n+m}F_{n+m}F_{n+2}$

(similarly derived as A)

3. By establishing the values of

$A' = 5[L_{4-m} - (-1)^m]$ and $B' = -(-1)^m(F_{m-4} - 1)$

we obtain:

m	A'
0	30
1	25
2	10
3	10
4	5
5	0
6	10

m	B'
0	4
1	1
2	2
3	0
4	1
5	0
6	0

4. By application of the the formula 1 and the values 3, we find:

A) $\underline{m = 0}$ $\quad \sqrt{(L_{2n} - L_{n-2}^2)\cdot(L_{2n+4} - L_n^2) + 30} = L_{2n+2} - L_{n-2}L$

$$= 5F_{2n} - 3(-1)^n$$

B)
$$\left.\begin{array}{ll} \underline{m = 3} & \sqrt{(F_{2n+3} - F_{n+1}^2)\cdot(F_{2n+7} - F_{n+3}^2)} = F_{2n+5} - F_{n+1}F_{n+3} \\ \underline{m = 5} & \sqrt{(F_{2n+5} - F_{n+3}^2)\cdot(F_{2n+9} - F_{n+5}^2)} = F_{2n+7} - F_{n+3}F_{n+5} \\ \underline{m = 6} & \sqrt{(F_{2n+6} - F_{n+4}^2)\cdot(F_{2n+10} - F_{n+6}^2)} = F_{2n+8} - F_{n+4}F_{n+6} \end{array}\right\}*$$

$*$ = all three versions identical to $-(-1)^m F_{n+2}F_{n+4}$,

which had to be shown.

5. Some special properties of the sequences (each sequence shows its own distinct characteristic, depending on m):

A)

m	Sequence
1	$5(F_{n-1}F_n + F_{n+1}^2)$, which implies: No integral divisor of any term is congruent to 3 or 7 modulo 10.
2, 6	sequences identical, but with phase difference
4	unique term for any n: $-3(-1)^n$

B)

1, 4, 7	$F_rF_{r+1} + F_{r+2}^2$, with phase differences. No integral divivor of any term is congruent to 3 or 7 modulo 10.
3, 5, 6	sequences identical (see 4B)
...	etc.

Also solved by P. Bruckman, L. Dresel, P. Filipponi, L. Kuipers, B. Prielipp, and H. Seiffert.

Sum Formula

H-368 (Corrected) *Proposed by Andreas N. Philippou, Patras, Greece (Vol. 22, no. 2, May 1984)*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\cdots+kn_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1,\ \ldots,\ n_k,\ r-1},\ n \geqslant 0$$

Show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell,r-1}^{(k)}, \; n \geqslant 0, \; r \geqslant 2. \qquad (*)$$

The problem includes as special cases ($r = 2$) the following:

For any fixed integer $k \geqslant 2$,

$$\sum_{\substack{n_1, \ldots, n_k \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + 1}{n_1, \; \ldots, \; n_k, \; 1} = \sum_{\ell=0}^{n} f_{\ell+1}^{(k)} f_{n+1-\ell}^{(k)}, \; n \geqslant 0, \qquad (A)$$

where $f_n^{(k)}$ are the Fibonacci numbers of order k [1], [2].

In particular, for $k = 2$,

$$\sum_{\ell=0}^{[n/2]} (n + 1 - \ell)\binom{n - \ell}{\ell} = \sum_{\ell=0}^{n} F_{\ell+1} F_{n+1-\ell}, \; n \geqslant 0. \qquad (A.1)$$

The problem also includes as a special case ($k = 1$, $r \geqslant 2$) the following:

$$\binom{n + r - 1}{r - 1} = \sum_{\ell=0}^{n} \binom{n - \ell + r - 2}{r - 2}, \; n \geqslant 0, \; r \geqslant 2. \qquad (B)$$

References

1. A. N. Philippou & A. A. Muwafi. "Waiting for the K^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly 20*, no. 1 (1982):28-32.
2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order k and Multinomial Coefficients." *The Fibonacci Quarterly 21*, no. 2 (1983):82-86.

Solution by the proposer.

For any fixed $x \in (0, \infty)$ and k and r as in the problem, set

$$f_{n+1,r}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + n_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \; \ldots, \; n_k, \; r - 1} x^{n_1 + \cdots + n_k}, \; n \geqslant 0. \qquad (1)$$

We shall establish the more general result

$$f_{n+1,r}^{(k)}(x) = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)}(x) f_{n+1-\ell,r-1}^{(k)}(x), \; n \geqslant 0, \; r \geqslant 2. \qquad (2)$$

To do so, we consider random variables $X_1, \ldots, X_r$ ($r \geqslant 2$) which are independent and identically distributed as $G_k(\cdot; p)$ ($0 < p < 1$) (see [3]). Then $X_1 + \cdots + X_r$ is distributed as $NB_k(\cdot; r, p)$ and $X_2 + \cdots + X_r$ is distributed as $NB_k(\cdot; r - 1, p)$ [3]. Therefore,

$$P[X_1 = \ell + k] = p^{\ell+k} \sum_{\substack{\ell_1, \ldots, \ell_k \ni \\ \ell_1 + 2\ell_2 + \cdots + k\ell_k = \ell}} \binom{\ell_1 + \cdots + \ell_k}{\ell_1, \; \ldots, \; \ell_k} \left(\frac{1 - p}{p}\right)^{\ell_1 + \cdots + \ell_k}, \; \ell \geqslant 0; \qquad (3)$$

$$P[X_1 + \cdots + X_r = n + kr]$$
$$= p^{n+kr} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \; \ldots, \; n_k, \; r - 1} \left(\frac{1 - p}{p}\right)^{n_1 + \cdots + n_k}, \; n \geqslant 0; \qquad (4)$$

and

$$P[X_2 + \cdots + X_r = n + k(r-1) - \ell]$$
$$= p^{n+k(r-1)-\ell} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - \ell}} \binom{n_1 + \cdots + n_k + r - 2}{n_1, \ldots, n_k, r-2} \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k}, \; n \geq \ell. \tag{5}$$

Next, for $n \geq 0$ and $n \geq 2$,

$$[X_1 + \cdots + X_r = n + kr] = \bigcup_{\ell=0}^{n} \{[X = \ell + k] \cap [X_2 + \cdots + X_r = n + k(r-1) - \ell]\}, \tag{6}$$

with

$$\{[X_1 = \ell + k] \cap [X_2 + \cdots + X_r = n + k(r-1) - \ell]\} \cap \{[X_1 = \ell' + k] \cap [X_2 + \cdots + X_r = n + k(r-1) - \ell']\} = \emptyset \quad (0 < \ell' \leq n),$$

and

$[X_1 = \ell + k]$,

$[X_2 + \cdots + X_r = n + k(r-1) - \ell]$ are independent events $(0 \leq \ell \leq n)$. Hence, for $n \geq 0$ and $r \geq 2$,

$$P[X_1 + \cdots + X_r = n + kr] = \sum_{\ell=0}^{n} P[X_1 = \ell + k]P[X_2 + \cdots + X_r = n + k(r-1) - \ell]. \tag{7}$$

Set $(1-p)/p = x$, so that $x \in (0, \infty)$. Then relation (7) implies (2), by means of (1) and (3)-(5). Q.E.D.

For $x = 1$, relation (2) shows the proposed problem. In order to appreciate its generality, it is instructive to note the special cases (A), (A.1), and (B). (A) follows from (*) for $r = 2$, by means of the definition of $f_{n+1,r}^{(k)}$ and the formula of [1] and [2]:

$$f_{n+1}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \; n \geq 0.$$

(A.1) follows directly from (A), and (B) is a simple consequence of (*) and the definition of $f_{n+1,r}^{(k)}$.

References

1. See Reference 1 above (p. 381).
2. See Reference 2 above (p. 381).
3. A. N. Philippou, C. Georghiou, & G. N. Philippou. "A Generalized Geometric Distribution and Some of Its Properties." *Statistics and Probability Letters 1* (1983):171-175.

Also solved by P. Bruckman.

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