# FIBONACCI AND LUCAS NUMBERS OF THE FORM $3z^2 \pm 1$

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## 1. INTRODUCTION

The purpose of this paper, which is a continuation of [1], is to report some results regarding the generalized Fibonacci and Lucas numbers of the form $3z^2 \pm \mu$.

In particular, we show for the Fibonacci and Lucas numbers that the following relations hold:

$F_m = 3z^2 + 1$ if and only if $m = \pm 1, 2, \pm 7$

$F_m = 3z^2 - 1$ if and only if $m = -2, \pm 3, \pm 5$

$L_m = 3z^2 + 1$ if and only if $m = 1, 3, 9$

$L_m = 3z^2 - 1$ if and only if $m = -1, 0, 5, \pm 8$

This author tried to show similar properties for other recursive sequences while working on class number problems for his Dissertation.

Throughout this paper we will make frequent use of the relations developed in [1]; thus, the numbering of the relations in this paper continues from that of [1].

Also, as in [1], $d$ will always be a rational integer of the first kind and $\frac{a + b\sqrt{d}}{2}$ will be the fundamental solution of the Pellian equation $x^2 - dy^2 = -4$. The sequences $\{U_m\}$ and $\{V_m\}$ are as defined in [1].

## 2. PRELIMINARIES

**Lemma 1:** i) Let $ab \not\equiv 0 \pmod 3$. Then the equation $U_m = 3z^2$ has

(a) the solutions $m = 0, 4$ if $d = 5$,
(b) only the solution $m = 0$ in all other cases.

ii) Let $b = 1$ and $a \not\equiv 0 \pmod 3$. Then the equation $U_m = 3z^2$ has

(a) the solutions $m = 0, 4$ if $d = 5$,
(b) the solutions $m = 0, 2$ if $a = 3z^2$,
(c) only the solution $m = 0$ in all other cases.

**Proof of i):** According to our assumptions $(U_m)_{m \in \mathbf{Z}}$ is periodic mod 3 with length of period 8 and 3 divides $U$ if and only if 4 divides $m$. Hence, $U_m = 3z^2$ implies $U_{2n}V_{2n} = 3z^2$, by (5). Since $n = 0$ is an obvious solution, we assume $n \neq 0$.

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Case 1. Let $n \not\equiv 0 \pmod 3$. Then $(U_{2n}, V_{2n}) = 1$, by (10), and we obtain

$$(U_{2n} = 3z_1^2,\ V_{2n} = z_2^2) \quad \text{or} \quad (U_{2n} = z_1^2,\ V_{2n} = 3z_1^2).$$

The first subcase is impossible, by (28). For the second, it is sufficient, by (30), to check only the value $n = 1$ in case $a$ and $b$ are both perfect squares ($n = 6$, $d = 5$, $L_{12} = 322 \neq 3z_2^2$).

For $n = 1$, we have

$$V_2 = a^2 + 2 = 3z_2^2,\ a = t^2,\ b = r^2.$$

That is, $t^4 + 2 = 3z_2^2$. Using [3], the last diophantine equation has at most one solution, $(t, z_2) = (\pm 1, \pm 1)$, which corresponds to the value $d = 5$.

Case 2. Let $n \equiv 0 \pmod 3$, $n \neq 0$. Equation (10) implies $(U_{2n}, V_{2n}) = 2$, so we must have

$$(U_{2n} = 2z_1^2,\ V_{2n} = 6z_2^2) \quad \text{or} \quad (U_{2n} = 6z_1^2,\ V_{2n} = 2z_2^2).$$

The first subcase is impossible because, by (31), the only possible value of $n$ for which $U_{2n} = 2z_1^2$ is $n = 3$ ($d = 5$) for which $L_6 = 18 \neq 6z_2^2$. The second subcase has, by (29) and direct computation, no solution for $n = \pm 3$, $d = 5, 29$. The proof of (ii) follows along the same lines as the proof of (i); hence, the details are omitted.

**Lemma 2:** Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2$ has the solutions $m = \pm 2$ if $a^2 + 2 = 3z^2$ and no solution in all other cases.

**Proof:** Since $a \not\equiv 0 \pmod 3$, $(V_m)_{m \in \mathbf{Z}}$ is periodic mod 3 with length of period 8 and 3 divides $V_m$ if and only if $m \equiv \pm 2 \pmod 8$.

Case 1. Let $m \equiv \pm 2 \pmod{16}$. Then, $a^2 + 2 = 3z^2$. The solutions of this equation are given in [4] by

$$3z + a\sqrt{3} = (3 + \sqrt{3})(2 + \sqrt{3})^n \quad \text{for } n = 0, 1, 2, \ldots .$$

If $m \neq \pm 2$, then (4) says we only have to consider the case $m \equiv 2 \pmod{16}$. We write $m = 2 + 2 \cdot 3^s \cdot n$ where $8|n$ and $3\nmid n$. Then, by (22), $V_m \equiv -V_2 \pmod{V_n}$. If $V_m = 3z^2$, we have $(3z)^2 \equiv -3V_2 \pmod{V_n}$ where $8|n$ and $3\nmid n$, which is impossible since $V_n \equiv 2 \pmod 3$, $(V_n, 3) = 1$, and $(V_m, V_2) = (2, V_2) = 1$ imply $(-3V_2/V_n) = 1$ by (33).

Case 2. Let $m \equiv \pm 6 \pmod{16}$. If $m = \pm 6$, then $a^6 + 6a^4 + 9a^2 + 2 = 3z^2$ or $(a^2 + 2)(a^4 + 4a^2 + 1) = 3z^2$ so that $c(c^2 - 3) = 3z$ where $c = a^2 + 2 \equiv 0 \pmod 3$ by our assumption on $a$. Since $(c, c^2 - 3) = 3$, we need only check the following two subcases:

$$\text{(i)} \quad c = 3z_1^2,\quad c^2 - 3 = (3z_2)^2,$$

$$\text{(ii)} \quad c = (3z_1)^2,\quad c^2 - 3 = 3z_2^2.$$

By (i) we have $3z_2 = \pm 1$, which is impossible. By (ii), $3^3 z_1^2 - 1 = z_2^2$, which is impossible mod 3. Now let $m \equiv 6 \pmod{16}$ with $m \neq 6$. We write

$$m = 6 + 2 \cdot 3^s \cdot n, \text{ where } 8|n,\ 3\nmid n.$$

Then, by (22), $V_m \equiv -V_6 \pmod{V_n}$. If $V_m = 3z^2$, we have

$$(3z)^2 \equiv -3V_6 \pmod{V_n} \text{ with } 8|n,\ 3\nmid n. \tag{62}$$

By using (13) repeatedly with (4), we obtain

$$2V_n = 2V_{8\lambda} \equiv \cdots \equiv \pm 2V_4 \pmod{V_6}. \tag{63}$$

We now note that $(V_6, V_4) = (V_4, V_2) = 1$ and $V_6 \equiv 2 \pmod 8$, since $a \equiv 1 \pmod 2$. However,

$$2V_4 \equiv -2V_2 \pmod{V_4} \tag{64}$$

and, by (22),

$$2V_4 \equiv -2V_0 \equiv -4 \pmod{V_2}. \tag{65}$$

Applying the Jacobi symbol, we now have:

$$\begin{aligned}
\left(\frac{-3V_6}{V_n}\right) &= (-1)\left(\frac{V_{6/2}}{V_n}\right) = -\left(\frac{V_n}{V_{6/2}}\right) \\
&= -\left(\frac{\pm V_4}{V_{6/2}}\right), && \text{by (63);} \\
&= -\left(\frac{V_{6/2}}{V_4}\right), && \text{since } V_6 \equiv 2 \pmod 8; \\
&= -\left(\frac{-2V_2}{V_4}\right), && \text{by (64);} \\
&= \left(\frac{V_2}{V_4}\right), && \text{by (19);} \\
&= -1, && \text{by (65).}
\end{aligned}$$

Therefore, (62) has no solution and the Lemma follows.

**Lemma 3**: For the generalized Fibonacci numbers $U_n$ the following identity holds:

$$U_{4n\pm1} = U_{2n}V_{2n\pm1} + b. \tag{66}$$

**Proof**: This is like the relation (36) of Lemma 2 in [1].

**Lemma 4**: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 6z^2$ has no solution.

**Proof**: Since $a$ is odd and $a \not\equiv 0 \pmod 3$, we have $a \equiv \pm 1 \pmod 6$ or $a^2 \equiv 1 \pmod{12}$. In this case the generalized Lucas numbers are periodic mod 6 with period 24 as are the usual Lucas numbers. Hence, it still holds that

$$V_m \equiv 0 \pmod 6 \text{ if and only if } m \equiv 6 \pmod{12},$$

and

$$V_m \equiv 18 \pmod{24} \text{ if } m \equiv 6 \pmod 2.$$

With $V_m = 6z^2$, we now have $z^2 \equiv 3 \pmod 4$, which has no solution.

## 3. FIBONACCI NUMBERS OF THE FORM $3z^2 \pm 1$

From now on $b$ will always be 1; that is, $d = a^2 + 4$.

**Theorem 1**: The equation $U_m = 3z^2 + 1$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = \pm 1, \pm 7$, if $d = 5$,
(b) the solutions $m = \pm 1, \pm 5$, if $d = 13$,
(c) only the solutions $m = \pm 1$ in all other cases.

**Proof:** For $m = 4n \pm 1$, (66) implies that $U_{2n}V_{2n\pm 1} = 3z^2$. If $n = 0$, then $U_{2n} = 0$, so that $z = 0$ is a solution which gives us $m = \pm 1$. Now assume that $n \neq 0$, then $U_{2n}V_{2n\pm 1} \neq 0$. Corollary 9 of [1] implies $(U_{2n}, V_{2n\pm 1}) = 1$. Hence, we must have

$$U_{2n} = 3z_1^2, \quad V_{2n\pm 1} = z_2^2, \tag{67}$$

or

$$U_{2n} = z_1^2, \quad V_{2n\pm 1} = 3z_2^2. \tag{68}$$

By using (28) and a direct computation of $U_{2n}$, we find that (67) has a solution for $m = 5$ if $d = 13$, and one for $m = 7$ if $d = 5$. By using (30), we find that the possible values of $n$ for (68) to have a solution are $n = 6$ if $d = 5$, and $n = 1$ if $a = t^2$.

When $n = 6$, $d = 5$, we have $L_{13} = 521 \neq 3z^2$ and $L_{11} = 199 \neq 3z^2$ so (68) has no solution in this case. When $n = 1$, $a = t^2$, we have $V_1 = 3z_2^2 = a$, which is impossible. Furthermore, $V_3 = a^3 + 3a = 3z_2^2$, which implies that $a^2 + 3 = 3w^2$ or $t^4 + 3 = 3w^2$ or $27t^4 + 1 = w^2$. The last equation, by [2], has no solutions, so (68) is impossible.

Note that $m = -5$, $d = 13$ and $m = -7$, $d = 5$ are also solutions, by (3).

**Theorem 1′:** The equation $U_m = 3z^2 - 1$, $m \equiv 1 \pmod 2$ has only the solutions

$$m = \pm 3,\ 15 \text{ if } a^2 + 2 = 3z^2$$

and no solutions in all other cases.

**Proof:** This follows the arguments of Theorem 1 by using (36), Corollary 9, (28) and (29) from [1].

**Theorem 2:** Let $a^2 + 2 = p$ where $p$ is a prime. The equation $U_m = 3z^2 + a$, $m \equiv 0$ (mod 2) has only the solution $m = 2$.

**Proof:** Case 1. Let $m = 4n$. By (38), we have $U_{2n+1}V_{2n-1} = 3z^2$. But, Lemma 3 of [1] implies $(U_{2n+1}, V_{2n-1}) = V_2 = p$, so the following possibilities must be checked:

$$U_{2n+1} = 3z_1^2, \quad V_{2n-1} = z_2^2 \tag{69}$$

$$U_{2n+1} = z_1^2, \quad V_{2n-1} = 3z_2^2 \tag{70}$$

$$U_{2n+1} = 3pz_1^2, \quad V_{2n-1} = pz_2^2 \tag{71}$$

$$U_{2n+1} = pz_1^2, \quad V_{2n-1} = 3pz_2^2 \tag{72}$$

Equation (69) has no solutions, since the possible values for which $V_{2n-1}$ is a perfect square are given by (28) in [1] and none of them gives a solution to $U_{2n+1} = 3z_1^2$.

Equation (70) has no solution either, because the values of $n$ for which $U_{2n+1} = z^2$ are $n = 0, -1$, which gives $V_{-1} = -a \neq 3z_2^2$ and $V_{-3} = -(a^3 + 3a) \neq 3z_2^2$.

If we write $2n - 1 = 4\lambda \pm 1$ and apply (13) repeatedly, we find that

$$2V_{2n-1} \equiv -2V_{4\lambda - 4\pm 1} \equiv \cdots \equiv \pm 2V_{\ 1} \pmod{V_2}.$$

Hence, if $V_{2\ -1} = pz_2^2 = V_2z_2^2$, we have $V_2$ divides $\pm 2V_{\pm 1}$ or $a^2 + 2$ divides $\pm 2a$, which is impossible. Hence, (71) has no solution.

If we write $2n + 1 = 4\lambda \pm 1$ and apply (13 repeatedly, we find that

$$2U_{2n+1} \equiv -2U_{4\lambda - 4\pm 1} \equiv \cdots \equiv \pm 2U_1 \pmod{V_2}.$$

Hence, if $U_{2n+1} = pz_1^2 = V_2 z_1^2$, we have $V_2 | \pm 2$, which is impossible. Thus, (72) has no solution.

Case 2. Let $m = 4n - 2$. Equation (40) implies $U_{2n-2}V_{2n} + a = 3z^2 + a$, so $U_{2n-2}V_{2n} = 3z^2$.

If $n = 1$, then $U_{2n-2} = 0$ and $z = 0$ which is a solution giving $m = 2$. When $n \neq 1$, then $U_{2n-2} \neq 0$. Recalling Lemma 3 of [1], we see that $(U_{2n-2}, V_{2n})$ divides $V_2 = p$. Hence, we must check the following four possibilities:

$$U_{2n-2} = 3z_1^2, \quad V_{2n} = z_2^2 \tag{73}$$

$$U_{2n-2} = z_1^2, \quad V_{2n} = 3z_2^2 \tag{74}$$

$$U_{2n-2} = 3pz_1^2, \quad V_{2n} = pz_2^2 \tag{75}$$

$$U_{2n-2} = pz_1^2, \quad V_{2n} = 3pz_2^2 \tag{76}$$

Equation (73) has no solution by (28).

The solutions of $U_{2n-2} = z_1^2$ are ($n = 7$, $d = 5$), ($n = 2$ if $a = t^2$), and $n = 1$. For $n = 7$, we have $L_{14} = 843 \neq 3z_2^2$. For $n = 2$, $V_4 \neq 3z_2^2$ by Lemma 2 if $a \not\equiv 0 \pmod 3$, while $V_4 = 3z_2^2$ if $a \equiv 0 \pmod 3$ is obviously impossible. Since $n = 1$ is also impossible, (74) has no solutions.

If $n \equiv 0 \pmod 2$, then we can see that $V_{2n} \neq pz_2^2$ by the same argument given for Case 1.

Now let $n \equiv 1 \pmod 2$, $n \neq 1$. Since $V_{-2n} = V_{2n}$, it is sufficient to consider only the case $n \equiv 1 \pmod 4$, that is, $2n \equiv 2 \pmod 8$. We write $2n = 2 + 2t \cdot 3^s$ with $4 | t$ and $3 \nmid t$ so that $V_{2n} \equiv -V_2 \pmod{V_t}$. Applying (13) repeatedly, and taking into account that $t = 4\lambda$, we obtain

$$2V_t \equiv \pm 2V_0 \equiv \pm 4 \pmod{V_2}, \text{ that is, } (V_t, V_2) = 1,$$

which implies $p \nmid V_t$. Hence, $V_{2n} = pz^2$ implies $(pz)^2 \equiv -p^2 \pmod{V_t}$, which is impossible since $(-p^2/V_t) = -1$ by (19). Therefore, (75) has no solution.

Now let $U_{2n-2} = pz_1^2$. Equation (5) implies that $U_{n-1}V_{n-1} = pz_1^2$. If $n \not\equiv 1 \pmod 3$, then $(U_{n-1}, V_{n-1}) = 1$ by (10), and we have

$$(U_{n-1} = pz_3^2,\ V_{n-1} = z_4^2) \quad \text{or} \quad (U_{n-1} = z_3^2,\ V_{n-1} = pz_4^2).$$

By using (28) and (30), we see that both are impossible.

If $n \equiv 1 \pmod 3$, then (10) implies $(U_{n-1}, V_{n-1}) = 2$, and we have

$$(U_{n-1} = 2pz_3^2,\ V_{n-1} = 2z_4^2) \quad \text{or} \quad (U_{n-1} = 2z_3^2,\ V_{n-1} = 2pz_4^2).$$

The first is impossible by (29) and a direct computation of $U_{n-1}$; the second is impossible by (31) and a direct computation of $V_{n-1}$. [For the second case, with $n = 4$, we should have $V_3 = 2pz_4^2$, which is impossible since, otherwise, we would have $p = a^2 + 2$ dividing $V_3 = a(a^2 + 3)$].

**Theorem 2′**: Let $a^2 + 2 = p$ where $p$ is a prime. The equation $U_m = 3z^2 - a$, $m \equiv 0 \pmod 2$ has

(a) the solutions $m = -2, 0, 6$, if $a = 3t^2$,
(b) only the solution $m = -2$ in all other cases.

**Proof**: The proof of this theorem follows that of Theorem 1′ with the exception of the case

$$U_{2n-1} = z_1^2, \quad V_{2n+1} = 3z_2^2, \text{ when } n = 1.$$

Under these conditions, we have $V_3 = 3z_2^2$, which can be transformed by simple reasoning into $27\mu^4 + 1 = v^2$, which has no solution by [2].

**Corollary 1:** (a) $F_m = 3z^2 + 1$ if and only if $m = \pm 1, 2, \pm 7$.

(b) $F_m = 3z^2 - 1$ if and only if $m = -2, \pm 3, \pm 5$.

## 4. LUCAS NUMBERS OF THE FORM $3z^2 \pm 1$

**Theorem 3:** Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2 + a$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = 1, 3, 9$ if $d = 5$,

(b) only the solution $m = 1$ in all other cases.

**Proof: Case 1.** Let $m = 4n - 1$. Equation (42) implies that $V_{2n-1}V_{2n} = 3z^2$. However, $(V_{2n-1}, V_{2n}) = 1$, so we have

$$(V_{2n-1} = z_1^2,\ V_{2n} = 3z_2^2) \quad \text{or} \quad (V_{2n-1} = 3z_1^2,\ V_{2n} = z_2^2).$$

For the first subcase, (28) implies

$$\begin{array}{lll} & n = 1 & \text{if } a = t^2,\ d \neq 5, \\ & n = 1, 2 & \text{if } \qquad d = 5, \\ \text{or} & n = 2 & \text{if } \qquad d = 13. \end{array}$$

When $n = 1$ and $a = t^2$, $V_{2n} = 3z_2^2$ if and only if $3z_2^2 - t^4 = 2$. Ljunggren [3] has proved that this equation possesses only the solution $(z_2, t) = (\pm 1, \pm 1)$, which gives $a = 1$ and so $d = 5$.

For $n = 2$, $d = 5$, we have $L_4 = 7 \neq 3z_2^2$, while for $n = 2$, $d = 13$, we obtain $L_{14} = 119 \neq 3z_2^2$.

By using (28) once more, we see that the second subcase has no solution.

**Case 2.** Let $m = 4n + 1 = 2(2n) + 1$. Equation (42) implies that $V_{2n}V_{2n+1} - 2a = 3z^2$. By (8) and (42), we see that

$$\{V_{2n}^2 - 2(-1)^n\}\{V\ V_{n+1} - (-1)^n a\} - 2a = 3z^2$$

or

$$(V_n^3 V_{n+1} - (-1)^n a V_n^2 - 2(-1)^n V_n V_{n+1} = 3z^2.$$

Hence, $V_n M_n = 3z^2$ with

$$M_n = V_n^2 V_{n+1} - (-1)^n a V_n - 2(-1)^n V_n.$$

Let $p$ be an odd prime not equal to 3 with $p^e \| V_n$. Since $p \nmid M_n$, we have $e \equiv 0 \pmod 2$. This implies that $V_n = w^2$ or $V_n = 2w^2$ or $V_n = 3w^2$ or $V_n = 6w^2$.

When $V_n = w^2$, (28) implies

$$\begin{array}{lll} & n = 1 & \text{if } a = t^2,\ d \neq 5, \\ & n = 1, 3 & \text{if } \qquad d = 5, \\ \text{or} & n = 3 & \text{if } \qquad d = 13. \end{array}$$

When $n = 1$ and $a = t^2$, we have $m = 5$. Hence, we must examine the equation

$$a^5 + 5a^3 + 5a = 3z^2 + a$$

for solutions. According to our assumptions, this equation can be written as

$$(a^2 + 2)^2 + a^2 = 3f^2.$$

However, $a^2 \equiv 1 \pmod{12}$ and $3f^2 \equiv 10 \pmod{12}$, so the equation is unsolvable.

By direct calculation, we can show that for all other possible values of $n$ no solutions exist.

Let $V_n = 2w^2$. Using (29) and direct calculation, we find that the unique solution in this case is $n = 0$ or $m = 1$.

Let $V_n = 3w^2$. In this case, Lemma 2 implies that solutions exist only for $n = \pm 2$ if $a^2 + 2 = 3w_1^2$.

When $n = -2$, we have $m = -7$. Since $V_{-7} < 0$, we know that $V_{-7} \neq 3z \;\; + a$. Hence, we have only to check the case for $n = 2$ or $m = 9$, that is, the possible solutions of the equation $a^9 + 9a^7 + 27a^5 + 30a^3 + 9a = 3z^2 + a$. Factoring, we have $a(a^2 + 2)(a^6 + 7a^4 + 13a^2 + 4) = 3z^2$ which, by replacing $a^2 + 2$ with $3w_1^2$, becomes $a(a^6 + 7a^4 + 13a^2 + 4) = w_2^2$.

However, $(a, a^6 + 7a^4 + 13a^2 + 4) = (a, 4) = 1$, so it follows that

$$a^6 + 7a^4 + 13a^2 + 4 = s^2 \quad \text{or} \quad (a^2 + 4)(a^4 + 3a^2 + 1) = s^2.$$

Now, the greatest common divisor tells us that

$$\begin{aligned}(a^2 + 4, a^4 + 3a^2 + 1) &= (a^2 + 4, (a^2 + 4) \;\; - 5(a^2 + 3))\\ &= (a^2 + 4, 5(a^2 + 3))\\ &= (a^2 + 4, 5) = 1 \quad \text{or} \quad 5.\end{aligned}$$

If $(a^2 + 4, a^4 + 3a^2 + 1) = 1$, it follows that $a^2 + 4 = \lambda^2$ with $a = t^2$. This implies $a = 0$, which is impossible since $a > 0$.

Now let $(a^2+4, a^4+3a^2+1) = 5$. Then $a^2 + 4 = 5\lambda_1^2$ and $a^4 + 3a^2 + 1 = 5\lambda_2^2$ with $a = t^2 \equiv 1 \pmod 6$. Recall that $t^4 - 5\lambda_1^2 = -4$ has the solutions $t = 1$ and $t = 2$ by (28). When $t = 1$, $a = 1$ and $d = 5$. When $t = 2$, $a = 4$, which is impossible since $a \equiv 1 \pmod 2$.

Therefore, in this case, we have only the solution $m = 9$, $d = 5$.

By Lemma 4, $V_n = 6w^2$ has no solutions.

Following the arguments of Theorem 3, we can also show

**Theorem 4**: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2 - a$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = -1, 5$ if $d = 5$,
(b) only the solution $m = -1$ in all other cases.

**Theorem 5**: The equation $L_m = 3z^2 + 1$, $m \equiv 0 \pmod 2$ has no solution.

**Proof**: **Case 1**. Let $m = 4n$. Equation (8) implies that $L_{2n}^2 = L_{4n} + 2$, which is the same as $3z^2 + 1 = L_{2n}^2 - 2$. Hence, $3(z^2 + 1) = L_{2n}^2$, so that $3 | L_{2n}$. Therefore, $2n \equiv 2 \pmod 4$ or $m \equiv 4 \pmod 8$. Since for even $m$, $L_{-m} = L_m$, it is sufficient to consider only the case $m \equiv 4(16)$.

If $m = 4$, then $L_4 = 7 \neq 3z^2 + 1$.

Let $m \neq 4$. We write $m = 4 + 2n3$ with $8 | n$, $3 \nmid n$. Then $V_m \equiv -V \pmod{V_n}$ by (22). If $V_m = 3z^2 + 1$, we have $(3z)^2 \equiv -24 \pmod{V_n}$, where $8 | n$ and $3 \nmid n$. Since for $8 | n$, $V_n \equiv 2 \pmod 3$, we can now apply the Jacobi symbol which is calculated to be $-1$, by (19) and (20). Hence, no solution exists.

**Case 2**. Let $m = 4n + 2$. Equation (8) gives $L_{2n+1}^2 = L_{4n+2} - 2$ or $L_{2n+1}^2 = 3z^2 - 1$. But $L_{2n+1}^2 - 5F_{2n+1}^2 = -4$ and so $5F_{2n+1}^2 = 3(z^2 + 1)$. This implies that

$3 | F_{2n+1}$, which is impossible since 3 divides $F_m$ if and only if 4 divides $m$. Hence, in this case also, there are no solutions.

**Theorem 6:** The equation $L_m = 3z^2 - 1$, $m \equiv 0 \pmod 2$ has only the solutions $m = 0, \pm 8$.

**Proof:** The proof is the same as that of Theorem 3, where we take into account the fact that $L_m \equiv -1 \pmod{23}$ if 16 divides $n$.

**Corollary 2:** (a) $L_m = 3z^2 + 1$ if and only if $m = 1, 3, 9$.

(b) $L_m = 3z^2 - 1$ if and only if $m = -1, 0, 5, 18$.

**Remark:** We can apply (26) and (27) as in [1] in order to obtain some statements about the solutions of diophantine equations of the form

$$DY^2 = AX^4 + BX^2 + C.$$

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Note: All the particular cases listed in (4.2) are referenced in Gould [1] except P. F. Byrd, "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," *The Fibonacci Quarterly 1*, no. 1 (1963):16-24.

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