GENERALIZED WYTHOFF NUMBERS FROM SIMULTANEOUS FIBONACCI REPRESENTATIONS

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(Submitted September 1982)

1. INTRODUCTION

In developing a Zeckendorf theorem for double-ended sequences, Hoggatt and Bicknell-Johnson [1] found a remarkable pattern arising from applying Klarner's theorem [2],[3] on simultaneous representations using Fibonacci numbers. Here we study the properties of the array generated, after first providing enough background information to make this paper self-contained. We shall show relationships with the Lucas numbers, the Wythoff pair sequences, and generalized Wythoff numbers [7].

David Klarner [2] has proved

Klarner's Theorem: Given nonnegative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \ldots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \quad B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1},$$

for $|k_i - k_j| \geqslant 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_i\}_{-\infty}^{\infty}$, $F_{i+1} = F_i + F_{i-1}$, $F_1 = 1$, $F_2 = 1$.

Thus, to represent a single integer $m > 0$, we merely solve

$$A = 0 = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}, \quad B = m = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$

which has a unique solution by Klarner's Theorem. A constructive method of solution is given in [3], and we will soon use this idea to generate a most interesting array.

We shall also need some properties of Wythoff pairs (a_n, b_n), which are formed by letting $a_1 = 1$ and taking a_n as the smallest positive integer not yet used, and letting $b_n = a_n + n$. Wythoff pairs have been discussed, among other sources, in [4], [5], [6], [7], and [8]. Early values are shown below.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_n:	1	3	4	6	8	9	11	12	14	16	17	19	21	22
b_n:	2	5	7	10	13	15	18	20	23	26	28	31	34	36

We list the following properties:

$$a_k + k = b_k \tag{1.1}$$

$$a_{b_n} = a_n + b_n \quad \text{and} \quad b_{b_n} = a_n + 2b_n \tag{1.2}$$

$$a_{a_n} = b_n - 1 \quad \text{and} \quad b_{a_n} = a_n + b_n - 1 \tag{1.3}$$

$$a_{k+1} - a_k = \begin{cases} 2, & k = a_n \\ 1, & k = b_n \end{cases} \tag{1.4}$$

$$b_{k+1} - b_k = \begin{cases} 3, & k = a_n \\ 2, & k = b_n \end{cases} \tag{1.5}$$

Further, (a_n, b_n) are related to the Fibonacci numbers in several ways, one being that, if $A = \{a_n\}$ and $B = \{b_n\}$, then A and B are the sets of positive integers for which the smallest Fibonacci number used in the unique Zeckendorf representation has respectively an even or an odd subscript [9].

Also, the Wythoff pairs are related to the Golden Section Ratio

$$\alpha = (1 + \sqrt{5})/2,$$

and recall that $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\beta = 1/\alpha$, as

$$a_n = [n\alpha], \qquad b_n = [n\alpha^2], \tag{1.6}$$

where $[x]$ is the greatest integer in x.

Lastly, we recall the generalized Wythoff numbers A_n, B_n, and C_n of [7] with beginning values

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n:	1	4	5	8	11	12	15	16	19	22	23	26	29	30
B_n:	3	7	10	14	18	21	25	28	32	36	39	43	47	50
C_n:	2	6	9	13	17	20	24	27	31	35	38	42	46	49

and the following properties useful in this paper:

$$A_n = 2a_n - n \tag{1.7}$$

$$B_n = a_n + 2n = b_n + n \tag{1.8}$$

$$C_n = a_n + 2n - 1 = b_n + n - 1 = a_{a_n} + n \tag{1.9}$$

$$C_n + 1 = B_n \quad \text{and} \quad C_n - 1 = A_{a_n} \tag{1.10}$$

$$A_{n+1} - A_n = \begin{cases} 1, & n = b_k \\ 3, & n = a_k \end{cases} \tag{1.11}$$

$$B_{n+1} - B_n = \begin{cases} 3, & n = b_k \\ 4, & n = a_k \end{cases} \tag{1.12}$$

$$C_{n+1} - C_n = \begin{cases} 3, & n = b_k \\ 4, & n = a_k \end{cases} \tag{1.13}$$

$$A_{a_n} = a_n + 2n - 2 \quad \text{and} \quad B_{a_n} = 3a_n + n - 1 \tag{1.14}$$

$$A_{a_{b_n}} = A_{b_{a_n}} + 1 = A_{b_{a_n}+1} \tag{1.15}$$

The sequences A_n, B_n, and C_n divide the positive integers into three disjoint subsets, classified by Zeckendorf representation using Lucas numbers [9].

2. AN ARRAY ARISING FROM KLARNER'S DUAL ZECKENDORF REPRESENTATION

Recall the Klarner dual Zeckendorf representation given in §1, where

$$\begin{cases} A = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} = 0 \\ B = F_{k_1} + F_{k_2} + \cdots + F_{k_r} = n, \end{cases} \tag{2.1}$$

where $n = 1, 2, 3, \ldots, |k_i - k_j| \geqslant 2, i \neq j$, and the Fibonacci number F_j comes from the double-ended sequence $\{F_j\}_{-\infty}^{\infty}$. The constructive method described in our earlier work [3] for solving for the subscripts k_j to represent A and B leads to a symbolic display with a generous sprinkling of Lucas numbers L_n ($L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$) and Wythoff pairs.

Here we use only two basic formulas,

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad 2F_n = F_{n+1} + F_{n-2}, \tag{2.2}$$

to push both right and left in forming successive lines of the array. The display is for expressions for B only; A is a translation of one space to the right. At each step, $B = n$ and $A = 0$.

The basic column centers under F_{-1}. We continue to add $F_{-1} = 1$ at each step, using the rules given in (2.2) to simplify the result. For example, for $n = 1$, we have $F_{-1} = 1$. For $n = 2$, $F_{-1} + F_{-1} = 2F_{-1} = F_0 + F_{-3} = 2$. For $n = 3$, $F_{-1} + F_0 + F_{-3}$ becomes $F_1 + F_{-3} = 1 + 2 = 3$. We display Table 2.1 on the following page.

Many patterns are discernible from Table 2.1. There are always the same number of successive entries in a given column. Under F_{-2} there are L_1; under F_{-3}, L_2; under F_{-4}, L_3; and under F_{-k}, L_{k+1} successive entries. The columns to the right of F_{-1} (under F_0, for instance) have $L_n \pm 1$ alternately successive entries, but the same number of successive entries always appears in a given column. Also, we notice that once we have all spaces cleared except the extreme edges in the pattern being built, we start again in the middle, as in lines 4, 8, 19, 48, ..., $L_{2k} + 1$, ...

Reading down the columns, we write the sequence of numbers first using that F_k is its representation. For example, the sequence of numbers using F_{-1} is 1, 4, 8, 11, 15, 19, ..., with first difference $\Delta_1 = 3$ and second difference $\Delta_2 = 4$. We want only the numbers first used when reading down the columns, so for F_{-3} we would use 2, 9, 20, 27, ..., and ignore 3, 4, 10, 11, 21, 22, ... We list sequences appearing beneath F_k in Table 2.1 along with first and second differences:

F_0:	2, 6, 9, 13, 17, 20, 24, 27, ...	$\Delta_1 = 3$, $\Delta_2 = 4$
F_1:	3, 10, 14, 21, 28, 32, 39, 43, ...	$\Delta_1 = 7$, $\Delta_2 = 4$
F_2:	5, 16, 23, 34, 45, 52, ...	$\Delta_1 = 11$, $\Delta_2 = 7$
F_3:	7, 25, 36, 54, 72, ...	$\Delta_1 = 18$, $\Delta_2 = 11$
F_4:	12, 41, 59, 88, ...	$\Delta_1 = 29$, $\Delta_2 = 18$
F_{-1}:	1, 4, 8, 11, 15, 19, 22, 26, ...	$\Delta_1 = 3$, $\Delta_2 = 4$
F_{-2}:	5, 12, 16, 23, 30, 34, 41, 45, ...	$\Delta_1 = 7$, $\Delta_2 = 4$

(continued)

Table 2.1 F_n Used To Represent B from Klarner's Theorem

Subscript n:

B	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
1									x								
2							x			x							
3							x				x						
4							x		x		x						
5					x			x				x					
6					x					x		x					
7					x								x				
8					x				x				x				
9					x		x			x			x				
10					x		x				x		x				
11					x		x		x		x		x				
12			x			x		x						x			
13			x			x				x				x			
14			x			x					x			x			
15			x			x			x		x			x			
16			x					x				x		x			
17			x							x		x		x			
18			x												x		
19			x						x						x		
20			x				x			x					x		
21			x				x				x				x		
22			x				x		x		x				x		
23			x		x			x				x			x		
24			x		x					x		x			x		
25			x		x								x		x		
26			x		x				x				x		x		
27			x		x		x			x			x		x		
28			x		x		x				x		x		x		
29			x		x		x		x		x		x		x		
30	x			x		x		x								x	
31	x			x		x				x						x	
32	x			x		x					x					x	
33	x			x		x			x		x					x	
34	x			x				x				x				x	
35	x			x						x		x				x	
36	x			x									x			x	
37	x			x					x				x			x	
38	x			x			x			x			x			x	
39	x			x			x				x		x			x	
40	x			x			x		x		x		x			x	
41	x					x		x						x		x	
42	x					x				x				x		x	
43	x					x					x			x		x	
44	x					x			x		x			x		x	
45	x							x				x		x		x	
46	x									x		x		x		x	
47	x																x
48	x								x								x
49	x						x			x							x
50	x						x				x						x
51	x						x		x		x						x
52	x				x			x				x					x
53	x				x					x		x					x
54	x				x								x				x
55	x				x				x				x				x
56	x				x		x			x			x				x
57	x				x		x				x		x				x
58	x				x		x		x		x		x				x
59	x		x			x		x						x			x
60	x		x			x				x				x			x

F_{-3}: 2, 9, 20, 27, 38, 49, ... $\Delta_1 = 7$, $\Delta_2 = 11$

F_{-4}: 12, 30, 41, 59, 77, 88, ... $\Delta_1 = 18$, $\Delta_2 = 11$

F_{-5}: 5, 23, 52, 70, 99, ... $\Delta_1 = 18$, $\Delta_2 = 29$

Surely the reader sees the Lucas numbers 1, 3, 4, 7, 11, 18, 29, ..., as the first and second differences. In the next section, we write formulas for each term in the sequences given, and find both Lucas numbers and the Wythoff pair numbers.

As a final observation, notice that the sequences associated with F_k when k is a negative odd integer have different behavior than all the others listed. For those sequences, $\Delta_2 > \Delta_1$, and successive differences follow the pattern $\Delta_1, \Delta_2, \Delta_1, \Delta_2, \Delta_2, \ldots$, while all the others have $\Delta_2 < \Delta_1$ and a pattern of successive differences that begins $\Delta_1, \Delta_2, \Delta_1, \Delta_1, \Delta_2, \ldots$.

3. LUCAS NUMBERS AND THE WYTHOFF PAIRS

We write the general term u_n for the sequence of numbers first using F_k in its representation as observed from Table 2.1 for $k \geqslant 0$.

F_0: $u_n = 2n + a_n - 1$

F_1: $u_n = n + 3a_n - 1$

F_2: $u_n = 3n + 4a_n - 2$

F_3: $u_n = 4n + 7a_n - 4$

F_4: $u_n = 7n + 11a_n - 6$

F_5: $u_n = 11n + 18a_n - 11$

F_6: $u_n = 18n + 29a_n - 17$

Again we see the Lucas numbers L_n, defined by

$$L_1 = 1,\ L_2 = 3,\ \text{and } L_{n+1} = L_n + L_{n-1}.$$

Observe that the last terms are either L_n or one less than L_n, and the pattern of general terms seems to be

$$F_k:\ u_n = L_k n + L_{k+1} a_n - [L_k - (1 + (-1)^k)/2],$$

where a_n is the first member of a Wythoff pair.

Theorem 3.1: The sequence of numbers first using F_k, $k \geqslant 0$, in its representation arising from Klarner's theorem is given by

$$F_k:\ u_n = nL_k + a_n L_{k+1} - [L_k - (1 + (-1)^k)/2].$$

Proof: From [8], all Fibonacci representations can be put in the form

$$u_n = (2n - 1 - a_n)\Delta_2 + (a_n - n)\Delta_1 + u_1, \tag{3.1}$$

where Δ_1 and Δ_2 are the first and second differences and u_1 is the beginning term of the sequence. By the method of generation of the array,

$$u_1 = \begin{cases} L_{k+1}, & k \text{ even} \\ L_{k+1} + 1, & k \text{ odd} \end{cases}$$

and $\Delta_2 = L_{k+2}$, $\Delta_1 = L_{k+3}$ for $k \geqslant 1$. Substitution of these values into (3.1) yields the result quite quickly.

For $k = 0$, we note that the sequence for F_0 can be written from (3.1) by letting $\Delta_1 = 3$, $\Delta_2 = 4$, and $u_1 = 2$.

The sequence of general terms for the sequences using F_k when k is negative gives us a different story. First, take k negative and even:

$$F_{-2}: \quad u_n = n + 3a_n + 1 \qquad \Delta_1 = 7,\ \Delta_2 = 4$$

$$F_{-4}: \quad u_n = 4n + 7a_n + 1 \qquad \Delta_1 = 18,\ \Delta_2 = 11$$

suggesting

$$F_{-k}: \quad u_n = nL_{k-1} + a_nL_k + 1 \qquad \Delta_1 = L_{k+2},\ \Delta_2 = L_{k+1}$$

When k is negative and odd, we let $m = n - 1$ and list

$$F_{-1}: \quad u_n = 2m + a_m + 1$$

$$F_{-3}: \quad u_n = 3m + 4a_m + 2$$

$$F_{-5}: \quad u_n = 7m + 11a_m + 5$$

suggesting

$$F_{-k}: \quad u_n = mL_{k-1} + a_mL_k + L_{k-2} + 1.$$

Theorem 3.2: The sequence of numbers first using F_{-k} in its representation is given by

(i) F_{-2j}: $u_n = nL_{2j-1} + a_nL_{2j} + 1$;

(ii) F_{-1}: $u_n = 2(n - 1) + a_{n-1} + 1$;

(iii) $F_{-(2j+1)}$, $j > 0$: $u_n = (n - 1)L_{2j} + a_{n-1}L_{2j+1} + L_{2j-1} + 1$.

Proof: (i) follows readily from (3.1) by taking

$$\Delta_1 = L_{2j+2},\ \Delta_2 = L_{2j+3}, \text{ and } u_1 = L_{2j+1} + 1.$$

(ii) is proved by mathematical induction. Note that (ii) is true for early values. Study the pattern of successive differences $\Delta_1 = 3$, $\Delta_2 = 4$, and by the rules for generation of the array, we have

$$u_{n+1} - u_n = \begin{cases} 3, & n - 1 = b_i \\ 4, & n - 1 = a_j \end{cases}$$

Assume $u_k = 2(k - 1) + a_{k-1} + 1$. Then, when $k - 1 = b_i$, (1.4) lets us write

$$\begin{aligned} u_{k+1} = 3 + u_k &= 3 + 2(k - 1) + a_{k-1} + 1 \\ &= 3 + 2(k - 1) + a_k \\ &= 2k + a_k + 1. \end{aligned}$$

When $k - 1 = a_j$, we again apply (1.4), and

$$\begin{aligned} u_{k+1} = 4 + u_k &= 4 + 2(k - 1) + a_{k-1} + 1 \\ &= 2k + (a_{k-1} + 2) + 1 \\ &= 2k + a_k + 1, \end{aligned}$$

so that u_{k+1} again has the form of (ii), establishing (ii) by mathematical induction.

The general case (iii) can be proved by mathematical induction in a similar way by using (1.4), if we take $\Delta_1 = L_{2j+2}$, $\Delta_2 = L_{2j+3}$, and $u_1 = L_{2j-1} + 1$. We again have Δ_1 when $n - 1 = b_i$ and Δ_2 when $n - 1 = a_j$.

Corollary 3.2: A second formula for the sequence of numbers first using F_{-k} in its representation is given by

$$F_{-2j}:\quad u_n = b_n L_{2j-1} + a_n L_{2j-2} + 1,\ j > 0;$$
$$F_{-1}:\quad u_n = n + b_{n-1}$$
$$F_{-(2j+1)}:\quad u_n = b_{n-1} L_{2j} + (a_{n-1} + 1) L_{2j-1} + 1,\ j > 0.$$

Proof: Change the form of the sequence for F_{2j} given in Theorem 3.2 by applying (1.1):

$$\begin{aligned} u_n = nL_{2j-1} + a_n L_{2j} + 1 &= nL_{2j-1} + a_n L_{2j-1} + a_n L_{2j-2} + 1 \\ &= b_n L_{2j-1} + a_n L_{2j-2} + 1. \end{aligned}$$

Again apply (1.1) to F_{-1}:

$$\begin{aligned} u_n = 2(n - 1) + a_{n-1} + 1 &= (n - 1) + (n - 1 + a_{n-1}) + 1 \\ &= n + b_{n-1}. \end{aligned}$$

The proof for $F_{-(2j+1)}$ is similar.

If we take k negative and odd, and apply (3.1) to write the terms of the sequences, we observe

$$F_{-1}:\quad u_n = 5n - a_n - 3$$
$$F_{-3}:\quad u_n = 15n - 4a_n - 9$$
$$F_{-5}:\quad u_n = 40n - 11a_n - 24$$

leading us to

Theorem 2.3: If k is odd and greater than 1, then the sequence of numbers first using F_{-k} in its representation arising from Klarner's Theorem is given by

$$F_{-k}:\quad u_n = 5nF_{k+1} - L_k a_n - 5F_k + 1.$$

Proof: Let $u_1 = L_{k-2} + 1$, $\Delta_2 = L_{k+2}$, and $\Delta_1 = L_{k+1}$ in (3.1) and simplify using $L_{k+2} + L_k = 5F_{k+1}$.

4. THE GENERALIZED WYTHOFF NUMBERS

The generalized Wythoff numbers A_n, B_n, and C_n of [7] provide another description of the general term of the sequences arising from using F_k in the representation from Klarner's theorem. Observe that, for F_0,

$$u_n = 2n + a_n - 1 = C_n$$

by (1.9). Each sequence we have generated is a subsequence of the sequence for A_n, B_n, or C_n. The sequences for F_0 and F_{-3} contain only C_i's, while the sequences for F_{2k+1} contain only B_i's, $k \geqslant 0$. All of the other sequences contain A_i's exclusively.

Theorem 4.1: The sequences arising from first using F_{2k+1}, $k \geqslant 0$, in the representation from Klarner's Theorem are

$$F_1:\quad u_n = B_{a_n}$$

$$F_3:\quad u_n = B_{b_{a_n}}$$

$$F_5:\quad u_n = B_{b_{b_{a_n}}}$$

$$F_{2k+1},\ k \geqslant 0:\quad u_n = B_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k}}$$

Proof: We simplify the form B_i to demonstrate that u_n has the form given by Theorem 3.1. For F_1, observe (1.14).

For F_3, we apply (1.8) and then (1.2) and (1.3) in sequence finishing with (1.1) to obtain

$$\begin{aligned} B_{b_{a_n}} &= a_{b_{a_n}} + 2b_{a_n} = (a_{a_n} + b_{a_n}) + 2b_{a_n} = (b_n - 1) + 3(a_n + b_n - 1) \\ &= 4b_n + 3a_n - 4 = 4(n + a_n) + 3a_n - 4 = 4n + 7a_n - 4. \end{aligned}$$

For F_5, we apply the same sequence of steps repeatedly to reduce the subscripted subscripts. For F_{2k+1}, the reduction of subscripted subscripts will always follow the same steps repeatedly. We show Lemma 4.1 to demonstrate one step of the subscript-reduction process and to show that we will end with the required form in terms of Lucas numbers.

Lemma 4.1: $$L_{i+1}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k}} + L_i \underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k} = L_{i+3}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}} + L_{i+2}\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}$$

Proof: Apply (1.2) followed by (1.1).

$$\begin{aligned} L_{i+1}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k}} + L_i \underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k} &= L_{i+1}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}} + L_{i+1}\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k} + L_i \underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k} \\ &= L_{i+1}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}} + L_{i+2}\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k} \\ &= L_{i+1}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}} + L_{i+2}\left(\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1} + a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}}\right) \\ &= L_{i+3}a_{\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1}} + L_{i+2}\underbrace{b_{b_{\ddots_{b_{a_n}}}}}_{k-1} \end{aligned}$$

Theorem 4.2: F_0: $u_n = C_n$, F_{-1}: $u_n = A_{a_{n-1}+1}$, and F_{-3}: $u_n = C_{a_{a_{n-1}+1}}$

Proof: The form for F_0 follows by comparing (1.9) and Theorem 3.1. For F_{-1}, we apply (1.11) and (1.14) and then compare with Theorem 3.2:

$$A_{a_{n-1}+1} = A_{a_{n-1}} + 3 = (a_{n-1} + 2(n - 1) - 2) + 3 = 2(n - 1) + a_{n-1} + 1$$

For F_{-3}, by (1.9) followed by (1.3), (1.4), and (1.5):

$$c_{a_{a_{n-1}+1}} = a_{a_{a_{n-1}+1}} + 2a_{a_{n-1}+1} - 1 = (b_{a_{n-1}+1} - 1) + 2a_{a_{n-1}+1} - 1$$

$$= (b_{a_{n-1}} + 3) - 1 + 2(a_{a_{n-1}} + 2) - 1 = b_{a_{n-1}} + 2a_{a_{n-1}} + 5$$

Next, use (1.3) finished by (1.1),

$$c_{a_{a_{n-1}+1}} = (a_{n-1} + b_{n-1} - 1) + 2(b_{n-1} - 1) + 5 = a_{n-1} + 2b_{n-1} + 2$$

$$= a_{n-1} + 3(a_{n-1} + (n-1)) + 2 = 3(n-1) + 4a_{n-1} + 2,$$

and compare with Theorem 3.2.

Theorem 4.3: F_2: $u_n = A_{a_{b_n}} = A_{b_{a_n}+1}$

F_4: $u_n = A_{b_{b_{a_n}}+1}$

F_{2k}, $k > 0$: $u_n = A_{\underbrace{b_{b_{\cdot_{\cdot_{\cdot_{b_{a_n}}}}}}}_{k}+1}$

Proof: For F_2, use (1.15) and (1.14), followed by (1.2) and (1.1):

$$A_{a_{b_n}} = a_{b_n} + 2b_n - 2 = (b_n + a_n) + 2b_n - 2$$

$$= 3(n + a_n) + a_n - 2 = 3n + 4a_n - 2$$

Then compare with u_n as given in Theorem 3.1.

For F_4, first apply (1.15) and then (1.7). After than, use (1.2) followed by (1.1) repeatedly to reduce the subscripted subscripts.

$$A_{b_{b_{a_n}}+1} = A_{b_{b_{a_n}}} + 1 = 2a_{b_{b_{a_n}}} - b_{b_{a_n}} + 1 = 2(a_{b_{a_n}} + b_{b_{a_n}}) - b_{b_{a_n}} + 1$$

$$= 2a_{b_{a_n}} + (a_{b_{a_n}} + b_{a_n}) + 1 = 3(a_{a_n} + b_{a_n}) + b_{a_n} + 1$$

$$= 3a_{a_n} + 4(a_{a_n} + a_n) + 1 = 7(b_n - 1) + 4a_n + 1$$

$$= 7(a_n + n) + 4a_n - 6 = 7n + 11a_n - 6$$

Now compare with Theorem 3.1.

For F_{2k}, the steps are always the same as for F_4, except for more repetitions.

Theorem 4.4: F_{-2}: $u_n = A_{b_n+1}$

F_{-4}: $u_n = A_{b_{b_n}+1}$

F_{-2k}, $k > 0$: $u_n = A_{\underbrace{b_{b_{\cdot_{\cdot_{\cdot_{b_n}}}}}}_{k}+1}$

Proof: Use (1.15) and (1.7). Then reduce the subscripted subscripts repeatedly by applying (1.2) followed by (1.1), and compare with Theorem 3.2. Because the proof is so much like that for F_4 and F_{2k} in Theorem 4.3, we show only F_{-2}.

$$A_{b_n+1} = A_{b_n} + 1 = 2a_{b_n} - b_n + 1 = 2(a_n + b_n) - b_n + 1$$
$$= 2a_n + (a_n + n) + 1 = 3a_n + n + 1.$$

Theorem 4.5: F_{-5}: $u_n = A_{a_{b_{a_{a_{n-1}+1}}}}$

F_{-7}: $u_n = A_{a_{a_{a_{b_{a_{a_{n-1}+1}}}}}}$

$F_{-(2k+1)}$, $k \geqslant 2$: $u_n = A_{\underbrace{a_{a_{\cdot_{\cdot_{a}}}}}_{2k-3}{}_{b_{a_{a_{n-1}+1}}}}$

Proof: In a manner similar to the proofs of Theorem 4.2 and Theorem 4.3, the subscripted subscripts can be painfully reduced, eventually, to match the form of Theorem 3.2. But, we almost have subscripted subscripts using the Wythoff pairs numbers a_n and b_n, except for the last subscript.

We apply results of [8]. Let $U = \{u_n\}_{n=1}^{\infty}$ be a sequence of integers. If U^* is a subsequence of U such that the general term is formed by subscripted subscripts taken from the Wythoff pair numbers, then we give each a-subscript weight 1 and each b-subscript weight 2. Then, U^* has first and second differences Δ_1^* and Δ_2^* given by

$$\Delta_2^* = F_{w+1}\Delta_2 + F_w\Delta_1 \quad \text{and} \quad \Delta_1^* = F_w\Delta_2 + F_{w-1}\Delta_1,$$

where w is the weight of the sequence and Δ_1 and Δ_2 are the first and second differences of U, the original sequence.

Notice that F_{-5} has weight 4 because the last subscript could be either a_i or b_j. $A_{a_{n-1}+1}$ is the original sequence, so we have $\Delta_1 = 3$, $\Delta_2 = 4$ because, by Theorem 4.2, we are looking at the sequence for F_{-1}. Then

$$\Delta_2^* = 4F_5 + 3F_4 = 4 \cdot 5 + 3 \cdot 3 = 29 = L_7$$
$$\Delta_1^* = 4F_4 + 3F_3 = 4 \cdot 3 + 3 \cdot 2 = 18 = L_6$$

where these are the known value for F_{-5}. Since we know u_1 for F_{-5}, we must have the same sequence.

For $F_{-(2k+1)}$, $k \geqslant 2$, the weight is $2k$, and

$$\Delta_2^* = 4F_{2k+1} + 3F_{2k} = L_{2k+3}$$
$$\Delta_1^* = 4F_{2k} + 3F_{2k-1} = L_{2k+2},$$

which we recognize from earlier sections.

Discussion: The weights for all of the other sequences for F_k are easier to calculate. For example, F_{2k} in Theorem 4.3 has weight $2k + 1$ and we can use A_n as the original sequence, with $\Delta_1 = 3$, $\Delta_2 = 1$, so that

$$\Delta_1^* = 3F_{2k+2} + F_{2k+1} = L_{2k+3} \quad \text{and} \quad \Delta_2^* = 3F_{2k+1} + F_{2k} = L_{2k+2}$$

which we recognize. From Theorem 4.1, the weight of F_{2k+1} is also $2k + 1$, and B_n gives the original sequence, so that $\Delta_1 = 4$, $\Delta_2 = 3$,

$$\Delta_1^* = 4F_{2k+2} + 3F_{2k+1} = L_{2k+4} \quad \text{and} \quad \Delta_2^* = 4F_{2k+1} + 3F_{2k} = L_{2k+3},$$

which again are known from earlier work.

Notice that we can use original sequences to relate all of the sequences of this paper to the sequences for F_0, F_1, F_{-1}, and F_{-2}, by looking at the next to last subscript in u_n. The original sequence related to $F_{-(2k+1)}$ then is

$$A_{a_{n-1}+1},$$

the sequence for F_{-1}. Even the sequence for F_{-3} is so related, because

$$C_{a_{n-1}+1} = A_{a_{n-1}+1} + 1.$$

Now, F_{2k+1} has original sequence B_{a_n}, which is F_1, while F_{-2k} goes with A_{b_n}, which gives F_{-2}. Lastly, F_{2k} has original sequence A_{a_n}, which is related to F_0, since $C_n = A_{a_n} + 1$.

Further, all of the sequences are related to the sequences for F_{-1}, F_0, or F_1. All of the sequences for F_{2k+1} are subsequences of B_n and thus are related to F_1; F_{-3} and F_0 have sequences that are subsequences of C_n. All of the other sequences are subsequences of A_n, making them related to the sequence for F_{-1}.

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