COMBINATORIAL PROOF FOR A SORTING PROBLEM IDENTITY

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1. In [2], L. Carlitz suggests that a combinatorial proof of the relation

$$H(m, n, p) - H(m-1, n, p) - H(m, n-1, p) - H(m, n, p-1) = \binom{m+n}{n}\binom{n+p}{p}\binom{p+m}{m} \tag{1.1}$$

might be interesting, where

$$H(m, n, p) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{j+k}{k} \cdot \binom{n-j+p-k}{p-k}\binom{p-k+i}{i}\binom{m-i+k}{k}. \tag{1.2}$$

We give such a proof.

By a lattice point is meant a point with integral coordinates. By a path is meant a minimal path via lattice points, taking unit horizontal and vertical steps. Unless stated otherwise, only nonnegative integers will be used.

2. To fix the idea, we first give the proof of Brock's original problem [1]; i.e., to show that

$$H(m, n) - H(m-1, n) - H(m, n-1) = \binom{m+n}{n}^2, \tag{2.1}$$

where

$$H(m, n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{n-j+i}{i}\binom{m-i+j}{j}. \tag{2.2}$$

By Figure 1, the number of paths from $(0, 0)$ to (m, n) via (i, j) and then from (m, n) to $(m+n, n+m)$ via $(m+n-j, n+i)$ is

$$\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{m-i+j}{j}\binom{n-j+i}{i}. \tag{2.3}$$

Summed over $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$, (2.3) gives $H(m, n)$. $H(m, n)$ counts all the paths from $(0, 0)$ to $(m+n, n+m)$ via (m, n), but the paths are counted more than once.

For given i and j, each path from $(0, 0)$ to $(m+n, n+m)$ via (m, n) that passes over the segment joining (i, j) and $(i+1, j)$ is counted at (i, j) and again at $(i+1, j)$. The same is true for each path from (m, n) to $(m+n, n+m)$ that passes over the segment joining $(m+n-j, n+i)$ and $(m+n-j, n+i+1)$. The number of such paths is

$$\binom{i+j}{j}\binom{m-i-1+n-j}{n-j}\binom{m-i-1+j}{j}\binom{n-j+i}{i} \tag{2.4}$$

Summed over $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n$, (2.4) gives $H(m-1, n)$.

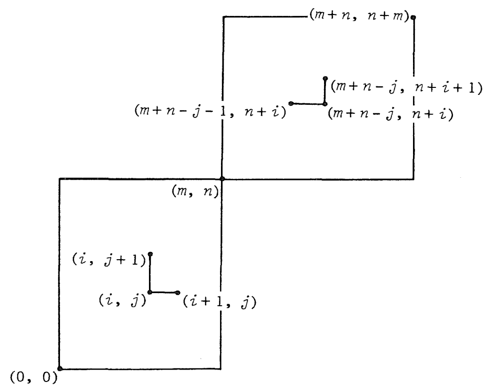

Figure 1

Similarly, $H(m, n-1)$ is obtained by considering those paths that pass over the segment joining (i, j) and $(i, j+1)$ and the segment joining $(m+n-j-1, n+i)$ and $(m+n-j, n+i)$ for $j = 0, 1, \ldots, n-1$. Alternatively, interchange m and n (and i and j) in the argument for $H(m-1, n)$.

Thus the left member of (2.1) counts each path from (0, 0) to $(m+n, n+m)$ via (m, n) exactly once. But the right member of (2.1) is just the number of such paths.

3. We now give the proof for (1.1).

By Figure 2, the number of paths from (0, 0) to (m, n) via (i, j), from (m, n) to $(m+n, n+p)$ via $(m+j, n+k)$, and from $(m+n, n+p)$ to $(m+n+p, n+p+m)$ via $(m+n+p-k, n+p+i)$ is

$$\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{j+k}{k}\binom{n-j+p-k}{p-k}\binom{p-k+i}{i}\binom{m-i+k}{k}. \tag{3.1}$$

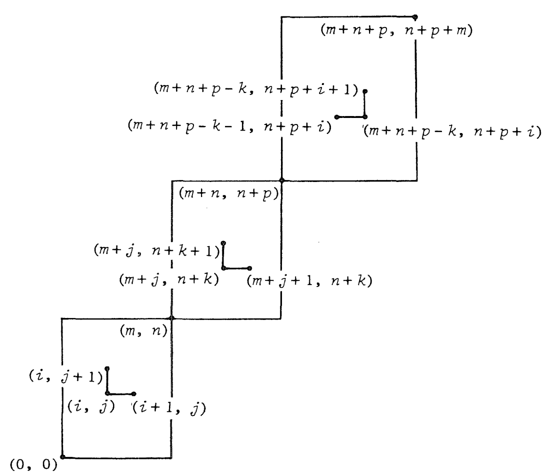

Figure 2

Sum (3.1) over $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$; and $k = 0, 1, \ldots, p$ to get $H(m, n, p)$. $H(m, n, p)$ counts all the paths from (0, 0) to $(m + n + p, n + p + m)$ via (m, n) and $(m + n, n + p)$. Again the paths are counted more than once.

For given i, j, k, each path that passes over the segment joining (i, j) and $(i + 1, j)$ is counted at (i, j) and again at $(i + 1, j)$. The same is true along the segment joining $(m + n + p - k, n + p + i)$ and $(m + n + p - k, n + p + i + 1)$. The number of such paths is

$$\binom{i + j}{j}\binom{m - i - 1 + n - j}{n - j}\binom{j + k}{k}\binom{n - j + p - k}{p - k}\binom{p - k + i}{i}\binom{m - i - 1 + k}{k} \tag{3.2}$$

Summing (3.2) over permissible values of i, j, and k, we get $H(m - 1, n, p)$.

$H(m, n - 1, p)$ is obtained by counting the paths that pass over the segment joining (i, j) and $(i, j + 1)$ and that joining $(m + j, n + k)$ and $(m + j + 1, n + k)$.

$H(m, n, p - 1)$ is obtained by counting the paths that pass over the segment joining $(m + j, n + k)$ and $(m + j, n + k + 1)$ and that joining $(m + n + p - k, n + p + i)$ and $(m + n + p - k - 1, n + p + i)$.

Thus, the left member of (1.1) counts each path from (0, 0) to $(m + n + p, n + p + m)$ via (m, n) and $(m + n, n + p)$. However, the right member of (1.1) is just that.

REFERENCES

1. Problem 60-2. *SIAM Review 4* 1962):396-398.
2. L. Carlitz. "A Binomial Identity Arising from a Sorting Problem." *SIAM Review 6* (1964):20-30.

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