ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.: Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-556 *Proposed by Valentina Bakinova, Rondout Valley, NY*

State and prove the general result illustrated by

$$4^2 = 16,\ 34^2 = 1156,\ 334^2 = 111556,\ 3334^2 = 11115556.$$

B-557 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there is no integer $n \geqslant 2$ such that

$$F_{3n-6}F_{3n-3}F_{3n+3}F_{3n+6} - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1985^8 + 1.$$

B-558 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there are no positive integers m and n such that

$$F_{4m}^2 - F_{3n} - 4 = 0.$$

B-559 *Proposed by László Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$. For positive integers n, prove that

$$[a + .5] + [a^2 + .5] + \cdots + [a^n + .5] = L_{n+2} - 2,$$

where $[x]$ denotes the greatest integer in x.

B-560 *Proposed by László Cseh, Cluj, Romania*

Let a and $[x]$ be as in B-559. Prove that

$$[aF_1 + .5] + [a^2F_2 + .5] + \cdots + [a^nF_n + .5]$$

is always a Fibonacci number.

B-561 *Proposed by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy*

(i) Let Q be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. For all integers n, show that

$$Q^n + (-1)^n Q^{-n} = L_n I, \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) Find a square root of Q, i.e., a matrix A with $A^2 = Q$.

SOLUTIONS

Double Product of 4 Consecutive Fibonacci Numbers

B-532 *Proposed by Herta T. Freitag, Roanoke, VA*

Find a, b, and c in terms of n so that

$$a^3(b - c) + b^3(c - a) + c^3(a - b) = 2F_nF_{n+1}F_{n+2}F_{n+3}.$$

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into

$$-(a - b)(b - c)(c - a)(a + b + c).$$

The equality is quickly verified when $a = F_{n+1}$, $b = F_{n+2}$, and $c = F_{n+3}$.

Also solved by Wray Brady, Paul S. Bruckman, L. Cseh, L. A. G. Dresel, L. Kuipers, I. Merényi, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

Product of 5 Fibonacci Numbers

B-533 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$g(a, b, c) = a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2).$$

Determine an infinitude of choices for a, b, and c such that $g(a, b, c)$ is the product of five Fibonacci numbers.

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into

$$-(b^2 - c^2)(c^2 - a^2)(a^2 - b^2).$$

With $a = F_n$, $b = F_{n+1}$, and $c = F_{n+2}$, this becomes $F_{n-1}F_nF_{n+2}F_{n+3}F_{2n}$.

Also solved by Wray Brady, Paul S. Bruckman, L. Cseh, L. A. G. Dresel, L. Kuipers, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

No Pythagorean Triangle with Square Area

B-534 *Proposed by A. B. Patel, India*

One obtains the lengths of the sides of a Pythagorean triangle by letting

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2,$$

where u and v are integers with $u > v > 0$. Prove that the area of such a triangle is not a perfect square when $u = F_{n+1}$, $v = F_n$, and $n \geqslant 2$.

I. Solution by L. A. G. Dresel, University of Reading, England

We have

$$a = (u - v)(u + v) = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2},$$

and the area is given by $A = \frac{1}{2}ab = F_{n-1}F_nF_{n+1}F_{n+2}$. Also,

$$F_{n-1}F_{n+2} = F_nF_{n+1} + (-1)^n.$$

It follows that the area A is the product of two consecutive integers, and thus cannot be a perfect square if $F_{n-1} > 0$, i.e., $n \geqslant 2$. In fact,

$$A = a(a + 1) \quad \text{when } n \text{ is odd,}$$

and

$$A = a(a - 1) \quad \text{when } n \text{ is even.}$$

II. Solution by L. Cseh (Cluj, Romania), J. M. Metzger (Grand Forks, ND), Bob Prielipp (Oshkosh, WI), Sahib Singh (Clarion, PA), and Lawrence Somer (Washington, D.C.), independently.

It is well known that the area of a Pythagorean triangle with integral sides is never a perfect square. For proof, see page 256 of *Elementary Number Theory* by David M. Burton (1980 edition). Thus, this result is true, in general, and therefore independent of involvement of Fibonacci numbers.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, K. Klauser, L. Kuipers, I. Merényi, M. Wachtel, Tad White, and the proposer.

Impossible Sum

B-535 *Proposed by L. Cseh & I. Merényi, Cluj, Romania*

Prove that there is no positive integer n for which

$$F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$$

Solution by L. A. G. Dresel, University of Reading, England

We have the identity $F_1 + F_2 + F_3 + \cdots + F_{3n} = F_{3n+2} - 1$, and it remains to prove that there is no integer n for which $F_{3n+2} - 1 = 16!$ If there were such an integer n, then, since Wilson's theorem gives

$$16! \equiv -1 \pmod{17}$$

we would require

$$F_{3n+2} \equiv 0 \pmod{17}.$$

Now, the first Fibonacci number divisible by 17 is $F_9 = 34$, and therefore F_m

is divisible by 17 if and only if 9 divides m. Clearly, there is no integer n for which $3n + 2$ is divisible by 9, and the result follows.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Diophantine Equation

B-536 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 2x^3 + 2x^2 + x + 1 = y^2.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

If we make the substitution

$$w = 1 + 2x + 2x^2, \tag{1}$$

the given equation is transformed to the simpler one,

$$4y^2 - w = 3. \tag{2}$$

Thus, $(2y - w)(2y + w) = 3$, which has only the four solutions:

$$(w, y) = (1, 1), (-1, 1), (-1, -1), (1, -1).$$

Setting $w = 1$ in (1) yields $2x(x + 1) = 0$, which implies $x = 0$ or $x = -1$. This yields four solutions (x, y), given by

$$(x, y) = (0, 1), (0, -1), (-1, 1), (-1, -1). \tag{3}$$

On the other hand, setting $w = -1$ in (1) yields $x^2 + x + 1 = 0$, which is impossible for integral x (the solutions being the complex cube roots of unity). Thus, *all* the integer solutions of the original equation are given by (3).

Also solved by L. Cseh, L. A. G. Dresel, Walther Janous, L. Kuipers, J. M. Metzger, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, and the proposer.

Another Diophantine Equation

B-537 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2.$$

Solution by John Oman & Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that the only solutions (x, y) in integers of the given equation are $(0, 1)$, $(0, -1)$, $(1, 3)$, $(1, -3)$, $(-1, 1)$, $(-1, -1)$, $(-3, 5)$, and $(-3, -5)$.

It is easily seen that (x, y) is a solution if and only if $(x, -y)$ is a solution. Hence, it suffices to find all solutions (x, y) in integers of the given equation where $y \geqslant 0$.

We begin with the following collection of equivalent equations:

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2$$

$$36x^4 + 108x^3 + 108x^2 + 36x + 36 = 36y^2$$

$$(6x^2 + 9x + 2)^2 + 3x^2 + 32 = (6y)^2 \quad (1)$$

$$(6x^2 + 9x + 3)^2 - 9(x + 3)(x - 1) = (6y)^2. \quad (2)$$

If $x > 1$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x > 1$. If $x < -3$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x < -3$. The problem is now easily completed.

Also solved by Paul S. Bruckman, L. Cseh, L. A. G. Dresel, Walther Janous, H. Klauser, J. M. Metzger, J. Suck, M. Wachtel, and the proposer.

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