THE SERIES OF PRIME SQUARE RECIPROCALS

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The series

$$\sum_p 1/p^2 \quad \text{(the sum being extended over all primes } p\text{)}$$

converges very slowly. Fortunately, the convergence can be quickened:

Lemma: $\sum_p 1/p^2 = \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log(\zeta(2k))$.

Proof: First,

$$\log(\zeta(2k)) = \log \prod_p (1 - (1/p^{2k}))^{-1}$$

by ([1], p. 246, Theorem 280).

$$\log \prod_p (1 - (1/p^{2k}))^{-1} = \sum_p - \log(1 - (1/p^{2k})) = \sum_p \sum_{s \geqslant 1} \frac{1}{sp^{2ks}}$$

$$= \sum_{s \geqslant 1} \frac{1}{s} \sum_p \frac{1}{p^{2ks}}.$$

We note the following for later use:

$$\log(\zeta(2k)) = \sum_{s \geqslant 1} \frac{1}{s} \sum_p \frac{1}{p^{2ks}}. \qquad (*)$$

Now,

$$\sum_{k \geqslant 1} \frac{\mu(k)}{k} \log(\zeta(2k)) = \sum_p \sum_{k \geqslant 1} \sum_{s \geqslant 1} \frac{\mu(k)}{ks} \cdot \frac{1}{p^{2ks}} = \sum_p \sum_{n \geqslant 1} \sum_{k|n} \mu(k) \cdot \frac{1}{np^{2n}}$$

$$= \sum_p \sum_{n \geqslant 1} \frac{1}{np^{2n}} \sum_{k|n} \mu(k) = \sum_p \frac{1}{p^2}$$

the last equality by ([1], p. 235, Theorem 263).

To compute $\sum_p 1/p^2$ accurately (to 28 decimal places), we calculate the first seven terms in the Lemma using exact values for the relevant arguments of the zeta function ζ (computed from [2], p. 298, Table 54, and p. 40, 2.), and we approximate the next twenty-four terms using (*). Thus, we obtain

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$$\sum_p \frac{1}{p^2} \approx \log\left(\frac{\pi^2}{6}\right) - \frac{1}{2}\log\left(\frac{\pi^4}{90}\right) - \frac{1}{3}\log\left(\frac{\pi^6}{945}\right) - \frac{1}{5}\log\left(\frac{\pi^{10}}{93555}\right) + \frac{1}{6}\log\left(\frac{691\pi^{12}}{638512875}\right)$$
$$- \frac{1}{7}\log\left(\frac{2\pi^{14}}{18243225}\right) + \frac{1}{10}\log\left(\frac{174611\pi^{20}}{1531329465290625}\right)$$
$$- \frac{1}{11}\Big\{(2^{-22} + 3^{-22} + 5^{-22} + \cdots + 23^{-22})$$
$$+ \frac{1}{2}(2^{-44} + 3^{-44}) + \frac{1}{3}(2^{-66}) + \frac{1}{4}(2^{-88})\Big\} \cdots$$
$$+ \frac{1}{46}(2^{-92}) - \frac{1}{47}(2^{-94}).$$

Our computer, when presented with this, answers:

$$\sum_p \frac{1}{p^2} \approx 0.4522474200410654985065433649.$$

It is easy to see that the Lemma holds not only for the exponent 2, but for all exponents $t > 1$. Hence,

$$(\forall\, t > 1) \sum_{k \geqslant 1} \frac{\pi(k) - \pi(k-1)}{k^t} - \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log(\zeta(tk)).$$

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. London: Oxford University Press, 1960.
2. Jahnke-Emde-Loesch. *Tables of Higher Functions*. 6th ed. New York: McGraw-Hill, 1960.

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